ALPHA CLASSES

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TARGET FORMULA BOOKLET

MATHEMATICS FOR

+1,+2 | IIT-JEE

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1-COMPLEX NUMBERS

- $\sqrt{-1}$ is denoted by 'i' and is pronounced as 'iota'.
  $i = \sqrt{-1} \Rightarrow i^2 = -1, i^3 = -i, i^4 = 1$.
- If $a, b \in R$ and $i = \sqrt{-1}$ then $a + ib$ is called a complex number. The complex number $a + ib$ is also denoted by the ordered pair (a, b)
- If $z = a + ib$ is a complex number, then :
  (i) a is called the real part of z and we write
  $\text{Re}(z) = a$.
  (ii) b is called the imaginary part of z and we write
  $\text{Im}(z) = b$
- Two complex numbers z_1 and z_2 are said to be equal complex numbers if $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.
- If $z = x + iy$ is a non zero complex number, then $1/z$ is called the multiplicative inverse of z.
- If $x + iy$ is a complex number, then the complex number $x - iy$ is called the conjugate of the complex number $x + iy$ and we write $\overline{x + iy} = x - iy$.
- **Algebra of Complex Numbers**
  (i) **Addition :** $(a + ib) + (c + id) = (a + c) + i(b + d)$
  (ii) **Subtraction :**
  $(a + ib) - (c + id) = (a - c) + i(b - d)$
  (iii) Multiplication :
  $(a + ib) + (c + id) = (ac - bd) + i(ab + bc)$
  (iv) **Division by a non-zero complex number :**
  $$\frac{a + ib}{c + id} = \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}, \ (c + id) \neq 0$$
- **Properties** : If z_1, z_2 are complex numbers, then
  (i) $\overline{(\bar{z}_1)} = z_1$
  (ii) $z + \bar{z} = 2\,\text{Re}(z)$
  (iii) $z - \bar{z} = 2i\,\text{Im}(z)$
  (iv) $z = \bar{z}$ iff z is purely real
  (v) $z = -\bar{z}$ iff z is purely imaginary
  (vi) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
  (vii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$
  (viii) $\overline{z_1 . z_2} = \bar{z}_1 . \bar{z}_2$
  (ix) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ provided $z_2 \neq 0$
- If $x + iy$ is a complex number, then the non-negative ral number $\sqrt{x^2 + y^2}$ is called the modulus of the complex number $x + iy$ and write
  $|x + iy| = \sqrt{x^2 + y^2}$
  Properties : If z_1, z_2 are complex numbers, then
  (i) $|z_1| = 0$ iff $z_1 = 0$
  (ii) $|z_1| = |\bar{z}_1| = |-z_1|$
  (iii) $-|z_1| \leq \text{Re}(z_1) \leq |z_1|$
  (iv) $-|z_1| \leq \text{Im}(z_1) \leq |z_1|$
  (v) $z_1 \bar{z}_1 = |z_1|^2$
  (vi) $|z_1 + z_2| \leq |z_1| + |z_2|$
  (vii) $|z_1 - z_2| \geq |z_1| - |z_2|$
  (viii) $|z_1 z_2| = |z_1| \, |z_2|$
  (ix) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, provided $z_2 \neq 0$
  (x) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1 \bar{z}_2)$
  (xi) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\text{Re}(z_1 \bar{z}_2)$
  (xi) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,[|z_1|^2 + |z_2|^2]$.
- De Moivre's Theorem
  (i) If n is any integer (positive or negative), then
  $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
  (ii) If n is a rational number, then the value or one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$
- Euler's Formula
  $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$
- Square root of complex number
  Square root of $z = a + ib$ are given by
  $$\pm\left[\sqrt{\left(\frac{|z| + a}{2}\right)} + i\sqrt{\left(\frac{|z| - a}{2}\right)}\right] \text{ for } b > 0 \text{ and}$$
  $$\pm\left[\sqrt{\left(\frac{|z| + a}{2}\right)} - i\sqrt{\left(\frac{|z| - a}{2}\right)}\right] \text{ for } b < 0.$$

- If $\omega = \frac{-1+i\sqrt{3}}{2}$, then the cube roots of unity are 1, ω and ω^2. We have:

  (i) $1 + \omega + \omega^2 = 0$ (ii) $\omega^3 = 1$

- Let z = x + iy be any complex number.

  Let z = r (cos θ + i sin θ) where r > 0.

  $\therefore$ x = r cos θ and y = r sin θ

  $\therefore x^2 + y^2 = r^2$

  $\Rightarrow \quad r = \sqrt{x^2+y^2} \qquad (\because r > 0)$

  $\therefore \cos\theta = \frac{x}{\sqrt{x^2+y^2}}$ and $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$

  The value of θ is found by solving these equations. θ is called the argument (or amplitude) of z.

  If – p < θ ≤ π, then θ is called the principal argument of z.

- Identification of θ –

x	y	arg(z)	Interval of θ
+	+	θ	$\left(0 < \theta < \frac{\pi}{2}\right)$
+	–	–θ	$\left(\frac{-\pi}{2} < \theta < 0\right)$
–	+	(π – θ)	$\left(\frac{\pi}{2} < \theta < \pi\right)$
–	–	–(π – θ)	$\left(-\pi < \theta < \frac{-\pi}{2}\right)$

- If z_1 and z_2 are two complex numbers then

  (i) $|z_1 - z_2|$ is the distance between the points with affixes z_1 and z_2.

  (ii) $\frac{mz_2 + nz_1}{m+n}$ is the affix of the point dividing the line joining the points with affixes z_1 and z_2 in the ratio m : n internally.

  (iii) $\frac{mz_2 - nz_1}{m-n}$ is the affix of the point dividing the line joining the points with affixes z_1 and z_2 in the ratio m : n externally where m ≠ n.

  (iv) If z_1, z_2, z_3 are the affixes of the vertices of a triangle then the affix of its centroid is $\frac{z_1+z_2+z_3}{3}$.

  (v) $z = tz_1 + (1 - t)z_2$ is the equation of the line joining points with affixes z_1 and z_2. Here 't' is a parameter.

  (vi) $\frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$ is the equation of the line joining points with affixes z_1 and z_2.

- Three points with affixes z_1, z_2, z_3 are collinear if

  $$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0.$$

- The general equation of a straight line is

  $\bar{a}z + a\bar{z} + b = 0$, where b is any real number.

- (i) $|z - z_1| < r$ represents the circle with centre z_1 and radius r.

  (ii) $|z - z_1| < r$ represents the interior of the circle with centre z_1 and radius r.

- $\left|\frac{z - z_1}{z - z_1}\right| = k$ represents a circle line which is the perpendicular bisector of the line segment joining points with affixes z_1 and z_2.

- $(z - z_1)\ (\bar{z} - \bar{z}_2) + (\bar{z} - \bar{z}_1) + (z - z_2) = 0$ represents the circle with line joining points with affixes z_1 and z_2 as a diameter.

- $|z - z_1| + |z - z_2| = 2k$, $k \in R^+$ represents the ellipse with foci at points with affixes z_1 and z_2.

- If z_1, z_2, z_3 be the affixes of the points A, B, C respectively, then the angle between AB and AC is given by $\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$.

- If z_1, z_2, z_3, z_4 are the affixes of the points A, B, C, D respectively, then the angle between AB and CD is given by $\arg\left(\frac{z_2 - z_1}{z_4 - z_3}\right)$.

- nth roots of a complex number

  Let z = r (cos θ + i sin θ), r > 0 be any complex number. nth root o $z = z^{1/n}$

  $$= r^{1/n}\left(\cos\frac{2k\pi + \theta}{n} + i\sin\frac{2k\pi + \theta}{n}\right),$$

  where k = 0, 1, 2, ………, n – 1.

  There are n distinct values and sum of all these values is 0.

- Logarithm of a complex number

  Let $z = re^{i\theta}$ be any complex number.

  Then $\log z = \log re^{i\theta} = \log r + \log e^{i\theta}$

  $= \log r + i\theta \log e = \log r + i\theta$.

  $\therefore \quad \log z = \log |z| + i \text{ amp } (z)$.

2-QUARDRATICS EQUATIONS

General quadratic equation :

An equation of the form

$ax^2 + bx + c = 0$ ...(1)

where $a \neq 0$, is called a quadratic equation, in the real or complex coefficients a, b and c.

Roots of a quadratic equation :

The values of x, (say x = α, β) which satisfy the quadratic equation (1) are called the roots of the equation and they are given by

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}\ ;\ \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Discriminant of a quadratic equation :

The quantity $D \equiv b^2 - 4ac$, is known as the discriminant of the equation.

Nature of the Roots :

In the equations $ax^2 + bx + c = 0$, let us suppose that a, b, c are real and $a \neq 0$. The following is true about the nature of its roots-

(i) The equation has real and distinct roots if and only if $D \equiv b^2 - 4ac > 0$.

(ii) The equation has real and coincident (equal) roots if and only if $D \equiv b^2 - 4ac = 0$.

(iii) The equation has complex roots of the form $\alpha \pm i\beta$, $\alpha \neq 0$, $\beta \neq 0 \in R$, if and only if $D \equiv b^2 - 4ac < 0$.

(iv) The equation has rational roots if and only if a, b, c ∈ Q (the set of rational numbers) and $D \equiv b^2 - 4ac$ is a perfect square (of a rational number).

(v) The equation has (unequal) irrational (surd form) roots if and only if $D \equiv b^2 - 4ac > 0$ and not a perfect square even if a, b and c are rational. In this case if $p + \sqrt{q}$, p, q rational, is an irrational root, then $P - \sqrt{q}$ is also a root (a, b, c being rational).

(vi) $\alpha + i\beta$ ($\beta \neq 0$ and $\alpha, \beta \in R$) is a root if and only if its conjugate $\alpha - i\beta$ is a root, that is complex roots occur in pairs in a quadratic equation. In case the equations is satisfied by more than two complex numbers, then it reduces to an identitiy.

$0.x^2 + 0.x + 0 = 0$, i.e. a = 0 = b = c.

Relation between Roots and Coefficients :

If α, β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then the sum and product of the roots is

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

Hence the quadratic equation whose roots are α and β is given by

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$ or $(x - \alpha)(x - \beta) = 0$

Condition that the two quadratic equations have a common root :

Let α be a common root of two quadratic equations

$a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$

where $a_1 \neq 0$, $a_2 \neq 0$ and $a_1b_2 - a_2b_1 \neq 0$.

Then $a_1\alpha^2 + b_1\alpha + c_1 = 0$ and $a_2\alpha^2 + b_2\alpha + c_2 = 0$

which gives (by cross multipication),

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

Thus eliminating α, the condition for a common root is given by

$(c_1a_2 - c_2a_1)^2 = (b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1)$ ...(2)

Condition that the two quadratic equations have both the roots common :

The two quadratic equation will have the same roots if and only if their coefficients are proportional, i.e.

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Descarte's rule of signs :

The maximum number of positive of a polynomial f(x) is the number of changes of signs in f(x) and the maximum number of negative roots of f(x) is the number of changes of signs in f(–x).

Position of roots :

If f(x) = 0 is an equation and a, b are two real numbers such that f(a) f(b) < 0, then the equation f(x) = 0 has at least one real root or an odd number of real roots between a and b. In case f(a) and f(b) are of the same sign, then either no real root or an even number of real roots of f(x) = 0 lie between a and b.

The quadratic expression :

(A) Let $f(x) \equiv ax^2 + bx + c$, $a, b, c \in R$, $a > d$ be a quadratic expression. Since,

$$f(x) = a\left\{\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right)\right\} \qquad ...(3)$$

The following is true from equation (3)

(i) $f(x) > 0$ (< 0) for all values of $x \in R$ if and only if $a > 0$ (< 0) and $D \equiv b^2 - 4ac < 0$.

(ii) $f(x) \geq 0$ (≤ 0) if and only if $a > 0$ (< 0) and $D \equiv b^2 - 4ac = 0$.

In this case ($D = 0$), $f(x) = 0$ if and only if $x = -\frac{b}{2a}$

(iii) If $D \equiv b^2 - 4ac > 0$ and $a > 0$ (< 0), then

$$f(x) = \begin{cases} < 0 (> 0), & \text{for x lying between the roots of } f(x) = 0 \\ > 0 (< 0), & \text{for x not lying between the roots of } f(x) = 0 \\ = 0, & \text{for x = each of the roots of } f(x) = 0 \end{cases}$$

(iv) If $a > 0$, (< 0), then $f(x)$ has a minimum (maximum) value at $x = -\frac{b}{2a}$ and this value is given by

$$[f(x)]_{\text{min (max)}} = \frac{4ac - b^2}{4a}$$

(B) **The sign of the expression :**

(i) The value of expression $(x - a)(x - b)$; $(a < b)$ is positive if $x < a$ or $x > b$, in other words x does not lie between a and b.

(ii) The expression $(x - a)(x - b)$; $(a < b)$ is negative if $a < x < b$ i.e. if x lies between a and b.

Some important results :

- If $f(\alpha) = 0$ and $f'(\alpha) = 0$, then α is a repeated root of the quadratic equation $f(x) = 0$ and $f(x) = a(x - \alpha)^2$. In fact $\alpha = -\frac{b}{2a}$.
- Imaginary and irrational roots occur in conjugate pairs (when a, b, c $\in R$ or a, b, c being rational) i.e., if $-3 + 2i$ or $5 - 2\sqrt{7}$ is a root then $-3 - 2i$ or $5 + 2\sqrt{7}$ will also be a root.
- For the quadratic equation $ax^2 + bx + c = 0$

  (i) One root will be reciprocal of the other if $a = c$.

  (ii) One root is zero if $c = 0$

  (iii) Roots are equal in magnitude but opposite in sign if $b = 0$.

  (iv) Both roots are zero if $b = c = 0$.

  (v) Roots are positive if a and c are of the same sign and b is of the opposite sign.

  (vi) Roots are of opposite sign if a and c are of opposite sign.

  (vii) Roots are negative if a, b, c are of the same sign.
- If the ratio of roots of the quadratic equation $ax^2 + bx + c = 0$ be p : q, then $pqb^2 = (p + q)^2ac$.
- If one root of the quadratic equation $ax^2 + bx + c = 0$ be p : q, then $pqb^2 = (p + q)^2ac$.
- If one root of the quadratic equation $ax^2 + bx + c= 0$ is equal to the n^{th} power of the other, then

  $$(ac^n)^{\frac{1}{n+1}} + (a^nc)^{\frac{1}{n+1}} + b = 0$$
- If one roots of the equation $ax^2 + bx + c = 0$ be n times the other root, then $nb^2 = ac(n + 1)^2$.
- If the roots of the equation $ax^2 + bx + c = 0$ are of the form $\frac{k+1}{k}$ and $\frac{k+2}{k+1}$, then $(a + b + c)^2 = b^2 - 4ac$.
- If the roots of $ax^2 + bx + c = 0$ are α, β, then the roots of $cx^2 + bx + a = 0$ will be $\frac{1}{\alpha}, \frac{1}{\beta}$.
- The roots of the equation $ax^2 + bx + c = 0$ are reciprocal to $a'x^2 + b'x + c' = 0$ if $(cc' - aa')^2 = (ba' - cb')(ab' - bc')$.
- Let $f(x) = ax^2 + bx + c$, where $a > 0$ .Then

  (i) Conditions for both the roots of $f(x) = 0$ to be greater than a given number K are $b^2 - 4ac \geq 0$; $f(K) > 0$; $\frac{-b}{2a} > K$.

  (ii) The number K lies between the roots of $f(x) = 0$ if $f(K) < 0$.'

  (iii) Condition for exactly one root of $f(x) = 0$ to lie between d and e is $f(d)\,f(e) < 0$.

3-PROGRESSION AND MATHEMATICAL INDUCTION

Arithmetic Progression (AP)

AP is a progression in which the difference between any two consecutive terms is constant. This constant difference is called **common difference** (c.d.) and generally it is denoted by d.

Standard form: Its standard form is

$a + (a + d) + (a + 2d) + \dots\dots\dots$

General term :

$T_n = a + (n - 1) d$

If $T_n = l$ then it should be noted that

(i) $d = \frac{\ell - a}{n - 1}$ (ii) $n = 1 + \frac{\ell - a}{d}$

Note: a, b, c are in AP $\Leftrightarrow 2b = a + c$

Sum of n terms of an AP :

$S_n = \frac{n}{2}(a + \ell)$

where l is last term (nth term). Replacing the value of l, it takes the form

$S_n = \frac{n}{2}[2a + (n - 1)d]$

Arithmetic Mean :

(i) If A be the AM between two numbers a and b, then $A = \frac{1}{2}(a + b)$

(ii) The AM of n numbers $a_1, a_2, \dots, a_n$

$= \frac{1}{n}(a_1 + a_2 + \dots + a_n)$

(iii) n AM's between two numbers

If $A_1, A_2, \dots, A_n$ be n AM's between a and b then

$a\ A_1, A_2, \dots, A_n, b$ is an AP of $(n + 2)$ terms. Its common difference d is given by

$T_{n+2} = b = a + (n + 1)d \Rightarrow d = \frac{b - a}{n + 1}$

so $A_1 = a + d, A_2 = a + 2d, \dots, A_n = a + nd$.

Sum of n AM's between a and b

$\therefore \Sigma A_n = n(A)$

Assuming numbers in AP :

(i) When number of terms be odd

Three terms : $a - d, a, a + d$

Five terms : $a - 2d, a - d, a, a + d, a + 2d$

....

(ii) When number of terms be even

Four terms: $a - 3d, a - d, a + d, a + 3d$

Six terms : $a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$

....

Geometrical Progression (GP) :

A progression is called a GP if the ratio of its each term to its previous term is always constant. This constant ratio is called its **common ratio** and it is generally denoted by r.

Standard form : Its standard form is

$a + ar + ar^2 + \dots\dots$

General term : $T_n = ar^{n-1}$

a, b, c are in GP $\Leftrightarrow \frac{b}{a} = \frac{c}{b} \Leftrightarrow b^2 = ac$

Sum of n terms of a GP :

The sum of n terms of a GP $a + ar + ar^2 + \dots\dots$ is given by

$$S_n = \begin{cases} \frac{a(1 - r^n)}{1 - r} = \frac{a - \ell r}{1 - r}, \text{when } r < 1 \\ \frac{a(r^n - 1)}{r - 1} = \frac{\ell r - a}{r - 1}, \text{when } r > 1 \end{cases}$$

when $\ell = T_n$.

Sum of an infinite GP :

(i) When $r > 1$, then $r^n \to \infty$, so $S_n \to \infty$ Thus when $r > 1$, the sum S of infinite GP $= \infty$

(ii) When $|r| < 1$, then $r^n \to 0$, so

$S = \frac{a}{1 - r}$

(iii) When $r = 1$, then each term is a so $S = \infty$.

Geometric Mean :

(i) If G be the GM between a and b then

$G = \sqrt{ab}$

(ii) G.M. of n numbers $a_1, a_2 \dots, a_n = (a_1a_2a_3 \dots a_n)^{1/n}$

(iii) n GM's between two numbers

$\Rightarrow r = (b/a)^{1/n+1}$

Product of n GM's between a and b

Product of GM's = $(ab)^{n/2} = G^n$

Assuming numbers in GP :

(i) When number of terms be odd

Three terms : a/r, a, ar

Five terms : a/r^2, a/r, a, ar, ar^2

....

(ii) When number of terms be even

Four terms : a/r^3, a/r, ar, ar^3

Six terms : a/r^5, a/r^3, a/r, ar, ar^3, ar^5

Arithmetic-Geometric Progression :

If each term of a progression is the product of the corresponding terms of an AP and a GP, then it is called arithmetic-geometric progression (AGP). For example:

a, (a + d)r, $(a + 2d)r^2$ ....

$T_n = [a + (n - 1)d]\, r^{n-1}$

$$S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r}$$

$$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2} \qquad |r| < 1$$

Harmonic Progression :

A progression is called a harmonic progression (HP) if the reciprocals of its terms are in AP.

Standard form : $\frac{1}{a} + \frac{1}{a+d} + \frac{1}{a+2d} + ...$

General term : $T_n = \frac{1}{a+(n-1)d}$

$\therefore$ a, b, c are in HP $\Leftrightarrow \frac{2}{b} = \frac{1}{a} + \frac{1}{c} \Leftrightarrow b = \frac{2ac}{a+c}$

Harmonic Mean :

(i) If H be a HM between two numbers a and b, then

$$H = \frac{2ab}{a+b} \text{ or } \frac{2}{H} = \frac{1}{a} + \frac{1}{b}$$

(ii) To find n HM's between a and b we first find n AM's between 1/a and 1/b, then their reciprocals will be the required HM's.

Relations between AM, GM and HM :

$G^2 = AH$

A > G > H, when a, b > 0.

If A and AM and GM respectively between two positive numbers, then those numbers are

$$A + \sqrt{A^2 - G^2}, A - \sqrt{A^2 - G^2}$$

Some Important Results :

- If number of terms in an AP/GP/HP is odd then its mid term is the AM/GM/HM between the first and last term.
- If number of terms in an AP/GP/HP is even the AM/GM/HM of its two middle terms is equal to the AM/GM/HM between the first and last term.
- a, b, c are in AP, GP and HP $\Leftrightarrow$ a = b = c
- a, b, c are in AP and HP $\Rightarrow$ a, b,c are in GP.
- a, b, c are in AP

  $\Leftrightarrow \frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in AP. $\Leftrightarrow$ bc, ca, ab are in HP.
- a, b, c are in GP $\Leftrightarrow a^2, b^2, c^2$ are in GP.
- a, b, c are in GP $\Leftrightarrow$ loga, logb, logc are in AP.
- a, b, c are in GP $\Leftrightarrow \log_a m\ \log_b m, \log_c m$ are in HP.
- a, b, c d are in GP $\Leftrightarrow$ a + b, b + c, c + d are in GP.
- a, b, c are in AP $\Leftrightarrow \alpha^a, \alpha^b, \alpha^c$ are in GP ($\alpha \in R_0$)

Principle of Mathematical Induction :

It states that any statement P(n) is true for all positive integral values of n if

(i) P(1) is true i.e., it is true for n = 1.

(ii) P(m) is true $\Rightarrow$ P(m + 1) is also true

i.e., if the statement is true for n = m then it must also be true for n = m + 1.

Some Formula based on the Principle of Induction :

- $\Sigma n = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

  (Sum of first n natural numbers)
- $\Sigma(2n - 1) = 1 + 3 + 5 + ... + (2n - 1) = n^2$

  (Sum of first n odd numbers)
- $\Sigma 2n = 2 + 4 + 6 + ... + 2n = n(n + 1)$

  (Sum of first n even numbers)
- $\Sigma n^2 = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$

  (Sum of the squares of first n natural numbers)
- $\Sigma n^3 = 1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$

  (Sum of the cubes of first n natural numbers)

Application in Solving Objective Question :

For solving objective question related to natural numbers we find out the correct alternative by negative examination of this principle. If the given statement is P(n), then by putting n = 1, 2, 3, ... in P(n), we decide the correct answer.

We also use the above formulae established by this principle to find the sum of n terms of a given series. For this we first express T_n as a polynomial in n and then for finding S_n, we put Σ before each term of this polynomial and then use above results of $\Sigma n, \Sigma n^2, \Sigma n^3$ etc.

4-BINOMIAL THEOREM

Binomial Theorem (For a positive Integral Index) :

If n is a positive integer and x, a are two real or complex quantities, then

$(x + a)^n = {}^nC_0\, x^n + {}^nC_1 x^{n-1}\, a + {}^nC_2\, x^{n-2}\, a^2 + \ldots + {}^nC_r x^{n-r}\, a^r + \ldots + {}^nC_{n-1} x\, a^{n-1} + {}^nC_n a^n$ ..(1)

The coefficient ${}^nC_0, {}^nC_1, \ldots\ldots, {}^nC_n$ are called binomial coefficients.

Properties of Binomial Expansion :

- There are (n + 1) terms in the expansion of $(x + a)^n$, n being a positive integer.
- In any term of expansion (1), the sum of the exponents of x and a is always constant = n.
- The binomial coefficients of term equidistant from the beginning and the end are equal, i.e. ${}^nC_r = {}^nC_{n-r}$ $(0 \le r \le n)$.
- The general term of the expansion is $(r + 1)^{th}$ term usually denoted by $T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$ $(0 \le r \le n)$.
- The middle term in the expansion of $(x + a)^n$

(a) If n is even then there is just one middle term, i.e. $\left(\frac{n}{2}+1\right)^{th}$ term.

(b) if n is odd, then there are two middle terms, i.e. $\left(\frac{n}{2}+1\right)^{th}$ term and $\left(\frac{n+3}{2}\right)^{th}$ term.

- The greatest term in the expansion of $(x + a)^n$, $x, a \in R$ and $x, a > 0$ can be obtained as below :

$$\therefore \quad \frac{T_{r+1}}{T_r} = \frac{n-r+1}{r}\frac{a}{x}$$

or $$\frac{T_{r+1}}{T_r} - 1 = \frac{(n+1)a - r(a+x)}{rx}$$

$$= \frac{(a+x)}{rx}\left\{\frac{(n+1)a}{a+x} - r\right\} = \frac{a+x}{rx}|k - r|,$$

where $k = \dfrac{(n+1)a}{a+x}$

Now, suppose that

(i) $k = \dfrac{(n+1)a}{a+x}$ is an integer. We have

$T_{r+1} > T_r \Leftrightarrow \dfrac{T_{r+1}}{T_r} > 1 \Leftrightarrow r < k$ (i.e. $1 \le r < k$)

Along , $T_{r+1} = T_r \Leftrightarrow \dfrac{T_{r+1}}{T_r} = 1 \Leftrightarrow r = k,$

i.e. $T_{k+1} = T_k > T_{k-1} > \ldots > T_3 > T_2 > T_1$

In this case there are two greatest terms T_k and T_{k+1}.

(ii) $k = \dfrac{(n+1)a}{a+x}$ is not an integer. Let [k] be the greatest integer in k. We have

$T_{r+1} > T_r \Leftrightarrow \dfrac{T_{r+1}}{T_r} \Leftrightarrow r < k = [k] + \text{(fraction)}$

$\Leftrightarrow r \le [k]$

i.e. $T_1 < T_2 < T_3 < \ldots < T_{[k]-1} < T_{[k]} < T_{[k]+1}$

In this case there is exactly one greatest term viz. $([k] + 1)^{th}$ term.

- **Term independent of** x in the expansion of $(x + a)^n$ – Let T_{r+1} be the term independent of x. Equate to zero the index of x and you will find the value of r.
- The number of term in the expansion of $(x + y + z)^n$ is $\dfrac{(n+1)(n+2)}{2}$, where n is a positive integer.

Pascal Triangle

In$(x + a)^n$ when expanded the various coefficients which occur are ${}^nC_0, {}^nC_1, {}^nC_2, \ldots$. The Pascal triangle gives the values of these coefficients for n = 0, 1, 2, 3, 4, 5, ...

n = 0	1
n = 1	1 1
n = 2	1 2 1
n = 3	1 3 3 1
n = 4	1 4 6 4 1
n = 5	1 5 10 10 5 1
n = 6	1 6 15 20 15 6 1
n = 7	1 7 21 35 35 21 7 1
n = 8	1 8 28 56 70 56 28 8 1

Rule : It is to be noted that the first and least terms in each row is 1. The terms equidistant from the beginning and end are equal. Any number in any row is obtained by adding the two numbers in the preceding row which are just at the left and just at the right of the given number, e.g. the number 21 in the row for n = 7 is the sum of 6 (left) and 15 (right) which occur in the preceding row for n = 6.

Important Cases of Binomial Expansion :

- If we put x = 1 in (1), we get
  $$(1 + a)^n = {}^nC_0 + {}^nC_1a + {}^nC_2a^2 + \dots\dots + {}^nC_ra^r + \dots\dots {}^nC_na^n \quad \dots(2)$$
- If we put x = 1 and replace a by – a, we get
  $$(1 - a)^n = {}^nC_0 - {}^nC_1a + {}^nC_2a^2 - \dots\dots + (-1)^r\, {}^nC_ra^r + \dots + (-1)^n\, {}^nC_na^n \quad \dots(3)$$
- Adding and subtracting (2) and (3), and then dividing by 2, we get
  $$\frac{1}{2}\{(1 + a)^n + (1 - a)^n\} = {}^nC_0 + {}^nC_2a^2 + {}^nC_4a^4 + \dots \quad \dots(4)$$
  $$\frac{1}{2}\{(1 + a)^n - (1 - a)^n\} = {}^nC_1a + {}^nC_3a^3 + {}^nC_5a^5 + \dots\dots \quad \dots(5)$$

Properties of Binomial Coefficients :

If we put a = 1 in (2) and (3), we get

$$2^n = {}^nC_0 + {}^nC_2 + \dots\dots + {}^nC_r + \dots + {}^nC_r + \dots\dots {}^nC_{n-1} + {}^nC_n$$

and $0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - \dots\dots + \dots\dots + (-1)^n\, {}^nC_n$

$$\therefore\ {}^nC_0 + {}^nC_2 + \dots = {}^nC_1 + {}^nC_3 + \dots = \frac{1}{2}[2^n \pm 0] = 2^{n-1} \quad \dots(6)$$

Due to convenience usually written as

$C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$

and $C_0 + C_1 + C_2 + C_3 + \dots\dots + C_n = 2^n$

$$\text{Where } {}^nC_r \equiv C_r = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

Some other properties to remember :

- $C_1 + 2C_2 + 3C_3 + \dots + nC_n = n \cdot 2^{n-1}$
- $C_1 - 2C_2 + 3C_3 - \dots = 0$
- $C_0 + 2C_1 + 3C_2 + \dots + (n + 1)C_n = (n + 2)\, 2^{n-1}$
- $C_0C_r + C_1C_{r+1} + \dots + C_{n-r}C_n = \dfrac{(2n)!}{(n-r)!.(n+r)!}$
- $C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 = \dfrac{(2n)!}{(n!)^2}$
- $C_0^2 - C_1^2 + C_2^2 - C_3^2 + \dots$
  $$= \begin{cases} 0, & \text{if n is odd} \\ (-1)^{n/2}.\,{}^nC_{n/2}, & \text{if n is even} \end{cases}$$

Binomial Theorem for Any Index :

- The binomial theorem for any index states that
  $$(1 + x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots \quad \dots(7)$$
  Where n is any index (positive or negative)
- The general term in expansion (7) is
  $$T_{r+1} = \frac{n(n-1)\dots\dots(n-r+1)}{r!}x^r$$
- In this expansion there are infinitely many terms.
- This expansion is valid for $|x| < 1$ and first term unity.
- When x is small compared with 1, we see that the terms finally get smaller and smaller. If x is very small compared with 1, we take 1 as a first approximation to the value of $(1 + x)^n$ or $1 + nx$ as a second approximation.
- Replacing n by – n in the above expansion, we get

$$(1 + x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \dots + (-1)^r \frac{n(n+1)(n+2)\dots(n+r-1)}{r!}x^r + \dots$$

Replacing x by – x in this expansion, we get

$$(1 - x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \dots\dots + \frac{n(n+1)(n+2)\dots(n+r-1)}{r!}x^r + \dots$$

Important expansions for n = –1, –2 are :

- $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \dots + (-1)^rx^r + \dots$ to ∞
- $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \dots + x^r + \dots$ to ∞
- $(1 + x)^{-2} = 1 - 2x + 3x^2 - \dots + (-1)^r(r + 1)x^r + \dots$ to ∞
- $(1 - x)^{-2} = 1 + 2x + 3x^2 + \dots + (r + 1)x^r + \dots$ to ∞
- $(1 + x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \dots + (-1)^r\dfrac{(r+1)(r+2)}{2!}x^r + \dots$
- $(1 - x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \dots + \dfrac{(r+1)(r+2)}{2!}x^r + \dots$

5-PERMUTATION AND COMBINATION

Permutation :

Definition : The ways of arranging or selecting a smaller or an equal number of persons or objects at a time from a given group of person or objects with due regard being paid to the order of arrangement or selection are called the (different) permutations.

Number of permutations without repetition :

- Arranging n objects, taken r at a time equivalent to filling r places from n things.

r-places :	1	2	3	4		r
Number of choice	n	(n–1)	(n–2)	(n–3)		n–(r – 1)

The number of ways of arranging = The number of ways of filling r places.

$= n(n-1)(n-2) \ldots\ldots. (n-r+1)$

$= \dfrac{n(n-1)(n-2)\ldots.(n-r+1)((n-r)!)}{(n-r)!} = \dfrac{n!}{(n-r)!}$

$= {}^{n}P_{r}$

- The number of arrangements of n different objects taken all at a time $= {}^{n}P_{n} = n!$

(i) ${}^{n}P_{0} = \dfrac{n!}{n!} = 1$; ${}^{n}P_{r} = n.\ {}^{n-1}P_{r-1}$

(ii) $0! = 1$; $\dfrac{1}{(-r)!} = 0$ or $(-r)! = \infty$ $(r \in N)$

Number of permutations with repetition :

- The number of permutations (arrangements) of n different objects, taken r at a time, when each object may occur once, twice, thrice, ... upto r times in any arrangement = The number of ways of filling r places where each place can be filled by any one of n objects.

r-places :	1	2	3	4		r
Number of choices :	n	(n)	(n)	(n)		n

The number of permutations = The number of ways of filling r places $= (n)^{r}$.

The number of arrangements that can be formed using n objects out of which p are identical (and of one kind) q are identical (and of another kind), r are identical (and of another kind) and the rest are distinct is $\dfrac{n!}{p!q!r!}$.

Condition permutations :

- Number of permutations of n dissimilar things taken r at a time when p particular things always occur $= {}^{n-p}C_{r-p}\ r!$.
- Number of permutations of n dissimilar things taken r at a time when p particular things never occur $= {}^{n-p}C_{r}\ r!$.
- The total number of permutations of n different things taken not more than r at a time, when each thing may be repeated any number of times, is $\dfrac{n(n^{r}-1)}{n-1}$.
- Number of permutations of n different things, taken all at a time, when m specified things always come together is $m! \times (n-m+1)!$.
- Number of permutation of n different things, taken all at a time, when m specified things never come together is $n! - m! \times (n-m+1)!$.
- Let there be n objects, of which m objects are alike of one kind, and the remaining (n – m) objects are alike of another kind. Then, the total number of mutually distinguishable permutations that can be formed from these objects is $\dfrac{n!}{(m!)\times(n-m)!}$.

The above theorem can be extended further i.e., if there are n objects, of which p_1 are alike of one kind; p_2 are alike of another kind; p_3 are alike of 3rd kind; .....; p_r are alike of r^{th} kind such that $p_1 + p_2 + \ldots. + p_r = n$; then the number of permutations of these n objects is $\dfrac{n!}{(p_1!)\times(p_2!)\times\ldots.\times(p_r!)}$.

Circular permutations :

Difference between clockwise and anti-clockwise arrangement : If anti-clockwise and clockwise order of arrangement are not distinct e.g., arrangement of beads in a necklace, arrangement of flowers in garland etc. then the number of circular permutations of n distinct items is $\dfrac{(n-1)!}{2}$.

- Number of circular permutations of n different things, taken r at a time, when clockwise and anticlockwise orders are taken as different is $\dfrac{{}^{n}p_{r}}{r}$.

- Number or circular permutations of n different things, taken r at a time, when clockwise and anticlockwise orders are not different is $\frac{^nP_r}{2r}$.

Theorems on circular permutations :

- **Theorem (i) :** The number of circular permutations on n different objects is (n – 1) !.
- **Theorem (ii) :** Then number of ways in which n persons can be seated round a table is (n – 1) !.
- **Theorem (iii) :** The number of ways in which n different beads can be arranged to form a necklace, is $\frac{1}{2}(n-1)!$.

Combinations :

Definition : Each of the different groups or selection which can be formed by taking some or all of a number of objects, irrespective of their arrangements, called a combination.

Notation : The number of all combinations of n things, taken r at a time is denoted by C(n, r) or ^nC_r or $\binom{n}{r}$. ^nC_r is always a natural number.

Difference between a permutation and combination :

- In a combination only selection is made whereas in a permutation not only a selection is made but also an arrangement in a definite order is considered.
- Each combination corresponds to many permutations. For example, the six permutations ABC, ACB, BCA, CBA and CAB correspond to the same combination ABC.

Number of combinations without repetition

The number of combinations (selections or groups) that can be formed from n different objects taken r $(0 \le r \le n)$ at a time is $^nC_r = \frac{n!}{r!(n-r)!}$. Also $^nC_r = {^nC_{n-r}}$.

Let the total number of selections (or groups) = x. Each group contains r objects, which can be arranged in r ! ways. Hence the number of arrangements of r objects = x × (r!). But the number of arrangements = ^np_r.

$\Rightarrow x(r\,!) = {^np_r} \Rightarrow x = \frac{n!}{r!(n-r)!} = {^nC_r}$

Number of combinations with repetition and all possible selections :

- The number of combinations of n distinct objects taken r at a time when any object may be repeated any number of times.
  = Coefficient of x^r in $(1 + x + x^2 + ... + x^r)^n$
  = Coefficient of x^r in $(1 - x)^{-n} = {^{n+r-1}C_r}$
- The total number of ways in which it is possible to form groups by taking some or all of n things at a time is $^nC_1 + {^nC_2} + ... + {^nC_n} = 2^n - 1$.
- The total number of ways in which it is possible to make groups by taking some or all out of $n = (n_1 + n_2 + ...)$ things, when n_1 are alike of one kind, n_2 are alike of second kind, and so on is $\{(n_1 + 1)(n_2 + 1) .....\} - 1$.
- The number of selections taking at least one out of $a_1 + a_2 + a_3 + ... + a_n + k$ objects, where a_1 are alike (of one kind), a_2 are alike (of second kind) and so on .... a_n are alike (of n^{th} kind) and k are distinct
  $= [(a_1 + 1)(a_2 + 1)(a_3 + 1) .... (a_n + 1)]2^k - 1$

Conditional combinations :

- The number of ways in which r objects can be selected from n different objects if k particular objects are
  (i) Always included = $^{n-k}C_{r-k}$
  (ii) Never included = $^{n-k}C_r$
- The number of combinations of n objects, of which p are identical, taken r at a time is
  $^{n-p}C_r + {^{n-p}C_{r-1}} + ... + {^{n-p}C_0}$, if $r \le p$ and
  $^{n-p}C_r + {^{n-p}C_{r-1}} + ... + {^{n-p}C_{r-p}}$, if $r > p$.

Division into groups

- The number of ways in which n different things can be arranged into r different groups is $^{n+r-1}P_n$ or n ! $^{n-1}C_{r-1}$ according as blank group are or are not admissible.
- Number of ways in which m × n different objects can be distributed equally among n persons (or numbered groups) = (number of ways of dividing into groups) × (number of groups)!
  $= \frac{(mn)!n!}{(m!)^n n!} = \frac{(mn)!}{(m!)^n}$
- If order of group is not important: The number of ways in which mn different things can be divided equally into m groups is $\frac{(mn)!}{(m!)^m m!}$.
- If order of groups is important : The number of ways in which mn different things can be divided equally into m distinct groups is
  $\frac{(mn)!}{(n!)^m m!} \times m\,! = \frac{(mn)!}{(n!)^m}$

Derangement :

Any change in the given order of the things is called a derangement.

If n things form an arrangement in a row, the number of ways in which they can be deranged so that no one of them occupies its original place is

$n\,!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+...+(-1)^n.\frac{1}{n!}\right)$

6-TRIGNOMETRIC RATIOS

Some Important Definitions and Formulae :

Measurement of angles : The angles are measured in degrees, grades or in radius which are defined as follows:

Degree : A right angle is divided into 90 equal parts and each part is called a degree. Thus a right angle is equal to 90 degrees. One degree is denoted by 1^o.

A degree is divided into sixty equal parts is called a minute. One minute is denoted by $1'$.

A minute is divided into sixty equal parts and each parts is called a second. One second is denoted by $1''$.

Thus,

1 right angle = 90^o (Read as 90 degrees)

$1^o = 60'$ (Read as 60 minutes)

$1' = 60''$ (Read as 60 seconds).

Grades : A right angle is divided into 100 equal parts and each part is called a grade. Thus a right angle is equal to 100 grades. One grade is denoted by 1^g.

A grade is divided into 100 equal parts and each part is called a minute and is denoted by $1'$.

A minute is divided into 100 equal parts and each part is called a second and is denoted by 1"

Thus,

1 right angled = 100^g (Read as 100 grades)

$1^g = 100'$ (Read as 100 minutes)

$1' = 100''$ (Read as 100 seconds)

Radians : A radian is the angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.

Domain and Range of a Trigono. Function :

If $f : X \to Y$ is a function, defined on the set X, then the domain of the function f, written as Domf is the set of all independent variables x, for which the image f(x) is well defined element of Y, called the co-domain of f.

Range of $f : X \to Y$ is the set of all images f(x) which belongs to Y, i.e.,

Range $f = \{f(x) \in Y : x \in X\} \subseteq Y$

The domain and range of trigonmetrical functions are tabulated as follows :

Trigo. Function	Domain	Range
sin x	R, the set of all the real number	$-1 \le \sin x \le 1$
cos x	R	$-1 \le \cos x \le 1$
tan x	$R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$	R
cosec x	$R - \{n\pi, n \in I\}$	$R - \{x : -1 < x < 1\}$
sec x	$R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$	$R - \{x : -1 < x < 1\}$
cot x	$R - \{n\pi, n \in I\}$	R

Relation between Trigonometrically Ratios and identities:

- $\tan\theta = \frac{\sin\theta}{\cos\theta}$; $\cot\theta = \frac{\cos\theta}{\sin\theta}$
- sin A cosec A = tan A cot A = cos A sec A = 1
- $\sin^2\theta + \cos^2\theta = 1$

  or $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$
- $1 + \tan^2\theta = \sec^2\theta$

  or $\sec^2\theta - \tan^2\theta = 1$ or $\sec^2\theta - 1 = \tan^2\theta$
- $1 + \cot^2\theta = \text{cosec}^2\theta$

  or $\text{cosec}^2\theta - \cot^2\theta = 1$ or $\text{cosec}^2\theta - 1 = \cot^2\theta$
- Since $\sin^2 A + \cos^2 A = 1$, hence each of sin A and cos A is numerically less than or equal to unity. i.e.

  $|\sin A| \le 1$ and $|\cos A| \le 1$

or $-1 \le \sin A \le 1$ and $-1 \le \cos A \le 1$

Note : The modulus of real number x is defined as $|x| = x$ if $x \ge 0$ and $|x| = -x$ if $x < 0$.

- Since sec A and cosec A are respectively reciprocals of cos A and sin A, therefore the values of sec A and cosec A are always numerically greater than or equal to unity i.e.

  sec A ≥ 1 or sec A ≤ -1

  and cosec A ≥ 1 or cosec A ≤ -1

  In other words, we never have

  $-1 <$ cosec A < 1 and $-1 <$ sec A < 1.

Trigonometrical Ratios for Various Angles :

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	–1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	–1	0	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	∞	0

Trigonometrical Ratios for Related Angles :

θ	$-\theta$	$\frac{\pi}{2}\pm\theta$	$\pi\pm\theta$	$\frac{3\pi}{2}\pm\theta$	$2\pi\pm\theta$
sin	$-\sin\theta$	$\cos\theta$	$\mp\sin\theta$	$-\cos\theta$	$\pm\sin\theta$
cos	$\cos\theta$	$\mp\sin\theta$	$-\cos\theta$	$\pm\sin\theta$	$\cos\theta$
tan	$-\tan\theta$	$\mp\cot\theta$	$\pm\tan\theta$	$\mp\cot\theta$	$\pm\tan\theta$
cot	$-\cot\theta$	$\mp\tan\theta$	$\pm\cot\theta$	$\mp\tan\theta$	$\pm\cot\theta$

Addition and Subtraction Formulae :

- $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$
- $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$
- $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$
- $\cot(A \pm B) = \frac{\cot A\cot B \mp 1}{\cot B \pm \cot A}$
- $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$ $= \cos^2 B - \cos^2 A$
- $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B$ $= \cos^2 B - \sin^2 A$

Formulae for Changing the Sum or Difference into Product :

- $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$
- $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$
- $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$
- $\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$

Formulae for Changing the Product into Sum or Difference :

- $2\sin A\cos B = \sin(A + B) + \sin(A - B)$
- $2\cos A\sin B = \sin(A + B) - \sin(A - B)$
- $2\cos A\cos B = \cos(A + B) + \cos(A - B)$
- $2\sin A\sin B = \cos(A - B) - \cos(A + B)$

Formulae Involving Double, Triple and Half Angles :

- $\sin 2\theta = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1+\tan^2\theta}$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

  $= 1 - 2\sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$
- $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$; $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$

  $\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$
- $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$
- $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

  or $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$
- $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

  or $\cos^3\theta = \frac{1}{4}(3\cos\theta + \cos 3\theta)$
- $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\left[\theta \neq n\pi + \frac{\pi}{6}\right]$

Trigonometrical Ratios for Some Special Angles :

θ	$7\frac{1^o}{2}$	15^o	$22\frac{1^o}{2}$
$\sin\theta$	$\frac{\sqrt{4-\sqrt{2}-\sqrt{6}}}{2\sqrt{2}}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$
$\cos\theta$	$\frac{\sqrt{4+\sqrt{2}+\sqrt{6}}}{2\sqrt{2}}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$
$\tan\theta$	$(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$	$2-\sqrt{3}$	$\sqrt{2}-1$

θ	18^o	36^o
$\sin\theta$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$
$\cos\theta$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{4}$
$\tan\theta$	$\frac{\sqrt{25-10\sqrt{5}}}{5}$	$\sqrt{5-2\sqrt{5}}$

Important Points to Remember :

- Maximum and minimum values of $a\sin x + b\cos x$ are $+\sqrt{a^2+b^2}$, $-\sqrt{a^2+b^2}$ respectively.

- $\sin^2 x + \operatorname{cosec}^2 x \geq 2$ for every real x.
- $\cos^2 x + \sec^2 x \geq 2$ for every real x.
- $\tan^2 x + \cot^2 x \geq 2$ for every real x
- If $x = \sec\theta + \tan\theta$, then $\frac{1}{x} = \sec\theta - \tan\theta$
- If $x = \operatorname{cosec}\theta + \cot\theta$, then $\frac{1}{x} = \operatorname{cosec}\theta - \cot\theta$
- $\cos\theta \cdot \cos 2\theta \cdot \cos 4\theta \cdot \cos 8\theta \;.... \cos 2^{n-1}\theta = \frac{\sin 2^n\theta}{2^n \sin\theta}$
- $\sin\theta \sin(60^\circ - \theta)\sin(60^\circ + \theta) = \frac{1}{4}\sin 3\theta$
- $\cos\theta \cos(60^\circ - \theta)\cos(60^\circ + \theta) = \frac{1}{4}\cos 3\theta$
- $\tan\theta \tan(60^\circ - \theta)\tan(60^\circ + \theta) = \tan 3\theta$

Conditional Identities :

1. If A + B + C = 180º, then

- $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$
- $\sin 2A + \sin 2B - \sin 2C = 4\cos A \cos B \sin C$
- $\sin(B + C - A) + \sin(C + A - B) + \sin(A + B - C) = 4\sin A \sin B \sin C$
- $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$
- $\cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A \sin B \cos C$

2. If A + B + C = 180º, then

- $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
- $\sin A + \sin B - \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
- $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
- $\cos A + \cos B - \cos C = -1 + 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$
- $\frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B} = 2$

3. If A + B + C = π, then

- $\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A \sin B \cos C$
- $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$
- $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A \cos B \cos C$
- $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2\sin A \sin B \cos C$

4. If A + B + C = π, then

- $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = 1 - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
- $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2 + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
- $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2} = 1 - 2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
- $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} - \cos^2\frac{C}{2} = 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

5. If x + y + z = π/2, then

- $\sin^2 x + \sin^2 y + \sin^2 z = 1 - 2\sin x \sin y \sin z$
- $\cos^2 x + \cos^2 y + \cos^2 z = 2 + 2\sin x \sin y \sin z$
- $\sin 2x + \sin 2y + \sin 2z = 4\cos x \cos y \cos z$

6. If A + B + C = π, then

- $\tan A + \tan B + \tan C = \tan A \tan B \tan C$
- $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$
- $\tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} + \tan\frac{A}{2}\tan\frac{B}{2} = 1$
- $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

7. (a) For any angles A, B, C we have

- $\sin(A + B + C)$
  $= \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$
- $\cos(A + B + C)$
  $= \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$
- $\tan(A + B + C)$
  $= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

(b) If A,B, C are the angles of a triangle, then

$\sin(A + B + C) = \sin\pi = 0$ and

$\cos(A + B + C) = \cos\pi = -1$

then (a) gives

$\sin A \sin B \sin C$
$= \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C$

and (a) gives

$1 + \cos A \cos B \cos C$
$= \cos A \sin B \sin C + \sin A \cos B \sin C + \sin A \sin B \cos C$

Method of Componendo and Dividendo :

If $\frac{p}{q} = \frac{a}{b}$, then by componendo and dividendo we can write

$\frac{p-q}{p+q} = \frac{a-b}{a+b}$ or $\frac{q-p}{q+p} = \frac{b-a}{b+a}$

or $\frac{p+q}{p-q} = \frac{a+b}{a-b}$ or $\frac{q+p}{q-p} = \frac{b+a}{b-a}$

7-TRIGNOMETRIC EQUATIONS

Functions with their Periods :

Function	Period
sin (ax + b), cos (ax + b), sec (ax + b), cosec (ax + b)	$2\pi/a$
tan(ax + b), cot (ax + b)	π/a
|sin (ax + b)|, |cos (ax + b)|, |sec (ax + b)|, |cosec (ax + b)|	π/a
|tan (ax + b)|, |cot (ax + b)|	$\pi/2a$

Trigonometrical Equations with their General Solution:

Trgonometrical equation	General Solution
$\sin\theta = 0$	$\theta = n\pi$
$\cos\theta = 0$	$\theta = n\pi + \pi/2$
$\tan\theta = 0$	$\theta = n\pi$
$\sin\theta = 1$	$\theta = 2n\pi + \pi/2$
$\cos\theta = 1$	$\theta = 2n\pi$
$\sin\theta = \sin\alpha$	$\theta = n\pi + (-1)^n \alpha$
$\cos\theta = \cos\alpha$	$\theta = 2n\pi \pm \alpha$
$\tan\theta = \tan\alpha$	$\theta = n\pi + \alpha$
$\sin^2\theta = \sin^2\alpha$	$\theta = n\pi \pm \alpha$
$\tan^2\theta = \tan^2\alpha$	$\theta = n\pi \pm \alpha$
$\cos^2\theta = \cos^2\alpha$	$\theta = n\pi \pm \alpha$
$\left.\begin{matrix}\sin\theta = \sin\alpha \\ \cos\theta = \cos\alpha\end{matrix}\right\|*$	$\theta = 2n\pi + \alpha$
$\left.\begin{matrix}\sin\theta = \sin\alpha \\ \tan\theta = \tan\alpha\end{matrix}\right\|*$	$\theta = 2n\pi + \alpha$
$\left.\begin{matrix}\tan\theta = \tan\alpha \\ \cos\theta = \cos\alpha\end{matrix}\right\|*$	$\theta = 2n\pi + \alpha$

* If α be the least positive value of θ which satisfy two given trigonometrical equations, then the general value of θ will be $2n\pi + \alpha$.

Note :

1. If while solving an equation we have to square it, then the roots found after squaring must be checked whether they satisfy the original equation or not. e.g. Let x = 3. Squaring, we get $x^2 = 9$, $\therefore$ x = 3 and – 3 but x = – 3 does not satisfy the original equation x = 3.

2. Any value of x which makes both R.H.S. and L.H.S. equal will be a root but the value of x for which $\infty = \infty$ will not be a solution as it is an indeterminate form.

3. If xy = xz, then x(y – z) = 0 $\Rightarrow$ either x = 0 or y = z or both. But $\frac{y}{x} = \frac{z}{x} \Rightarrow y = z$ only and not x = 0, as it will make $\infty = \infty$. Similarly, if ay = az, then it will also imply y = z only as $a \neq 0$ being a constant.

Similarly, $x + y = x + z \Rightarrow y = z$ and $x - y = x - z \Rightarrow y = z$. Here we do not take x = 0 as in the above because x is an additive factor and not multiplicative factor.

4. When $\cos\theta = 0$, then $\sin\theta = 1$ or –1. We have to verify which value of $\sin\theta$ is to be chosen which satisfies the equation $\cos\theta = 0 \Rightarrow \theta = \left(n + \frac{1}{2}\right)\pi$

If $\sin\theta = 1$, then obviously n = even. But if $\sin\theta = -1$, then n = odd.

Similarly, when $\sin\theta = 0$, then $\theta = n\pi$ and $\cos\theta = 1$ or –1.

If $\cos\theta = 1$, then n is even and if $\cos\theta = -1$, then n is odd.

5. The equations $a\cos\theta \pm b\sin\theta = c$ are solved as follows :

Put $a = r\cos\alpha$, $b = r\sin\alpha$ so that $r = \sqrt{a^2 + b^2}$ and $\alpha = \tan^{-1} b/a$.

The given equation becomes

$r[\cos\theta\cos\alpha \pm \sin\theta\sin\alpha] = c$;

$\cos(\theta \pm \alpha) = \frac{c}{r}$ provided $\left|\frac{c}{r}\right| \leq 1$.

Relation between the sides and the angle of a triangle:

1. **Sine formula :**

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{1}{2R}$$

Where R is the radius of circumcircle of triangle ABC.

2. **Cosine formulae :**

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \cos B = \frac{a^2 + c^2 - b^2}{2ac},$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

It should be remembered that, in a triangle ABC

- If $\angle A = 60^\circ$, then $b^2 + c^2 - a^2 = bc$
- If $\angle B = 60^\circ$, then $a^2 + c^2 - b^2 = ac$
- If $\angle C = 60^\circ$, then $a^2 + b^2 - c^2 = ab$

3. **Projection formulae :**

$a = b \cos C + c \cos B$, $b = c \cos A + a \cos C$

$c = a \cos B + b \cos A$

Trigonometrical Ratios of the Half Angles of a Triangle:

If $s = \frac{a+b+c}{2}$ in triangle ABC, where a, b and c are the lengths of sides of ΔABC, then

(a) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$, $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}$,

$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(b) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $\sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}$,

$\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(c) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$,

$\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$, $\tan\frac{C}{2}\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Napier's Analogy :

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}, \tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$$

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$$

Area of Triangle :

$$\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$$

$$\Delta = \frac{1}{2}\frac{a^2\sin B\sin C}{\sin(B+C)} = \frac{1}{2}\frac{b^2\sin C\sin A}{\sin(C+A)} = \frac{1}{2}\frac{c^2\sin A\sin B}{\sin(A+B)}$$

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc}$$

Similarly $\sin B = \frac{2\Delta}{ca}$ & $\sin C = \frac{2\Delta}{ab}$

Some Important Results :

1. $\tan\frac{A}{2}\tan\frac{B}{2} = \frac{s-c}{s}$ $\therefore \cot\frac{A}{2}\cot\frac{B}{2} = \frac{s}{s-c}$

2. $\tan\frac{A}{2} + \tan\frac{B}{2} = \frac{c}{s}\cot\frac{C}{2} = \frac{c}{\Delta}(s-c)$

3. $\tan\frac{A}{2} - \tan\frac{B}{2} = \frac{a-b}{\Delta}(s-c)$

4. $\cot\frac{A}{2} + \cot\frac{B}{2} = \dfrac{\tan\frac{A}{2} + \tan\frac{B}{2}}{\tan\frac{A}{2}\tan\frac{B}{2}} = \frac{c}{s-c}\cot\frac{C}{2}$

5. Also note the following identities :

- $\Sigma(p-q) = (p-q) + (q-r) + (r-p) = 0$
- $\Sigma p(q-r) = p(q-r) + q(r-p) + r(p-q) = 0$
- $\Sigma(p+a)(q-r) = \Sigma p(q-r) + a\Sigma(q-r) = 0$

Solution of Triangles :

1. **Introduction :** In a triangle, there are six elements viz. three sides and three angles. In plane geometry we have done that if three of the elements are given, at least one of which must be a side, then the other three elements can be uniquely determined. The procedure of determining unknown elements from the known elements is called solving a triangle.

2. **Solution of a right angled triangle :**

Case I. When two sides are given : Let the triangle be right angled at C. Then we can determine the remaining elements as given in the following table.

Given	**Required**
(i) a, b	$\tan A = \frac{a}{b}$, $B = 90^\circ - A$, $c = \frac{a}{\sin A}$
(ii) a, c	$\sin A = \frac{a}{c}$, $b = c\cos A$, $B = 90^\circ - A$

Case II. When a side and an acute angle are given – In this case, we can determine

Given	**Required**
(i) a, A	$B = 90^\circ - A$, $b = a\cot A$, $c = \frac{a}{\sin A}$
(ii) c, A	$B = 90^\circ - A$, $a = c\sin A$, $b = c\cos A$

8-STRAIGHT LINES AND CIRCLES

Different standard form of the equation of a straight line :

- **General form :** $Ax + By + C = 0$
  where A, B, C are any real numbers not all zero.
- **Gradient (Tangent) form :** $y = mx + c$
  It is the equation of a straight line which cuts off an intercept c on y-axis and makes an angle with the positive direction (anticlockwise) of x-axis such that $\tan\theta = m$. The number m is called slope or the gradient of this line.
- **Intercept form :**
  $$\frac{x}{a} + \frac{y}{b} = 1$$
  It is the equation of straight line which cuts off intercepts a and b on the axis of x and y respectively.
- **Normal form (Perpendicular form) :**
  $x\cos\alpha + y\sin\alpha = p$
  It is the equation of a straight line on which the length of the perpendicular from the origin is p and α is the angle which , this perpendicular makes with the positive direction of x-axis.
- **One point form :**
  $y - y_1 = m(x - x_1)$
  It is the equation of a straight line passing through a given point (x_1, y_1) and having slope m.
- **Parametric equation :**
  $$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$
  It is the equation of a straight line passes through a given point $A(x_1, y_1)$ and makes an angle θ with x-axis.
- **Two points form :**
  $$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$
  It is the equation of a straight line passing through two given points (x_1, y_1) and (x_2, y_2), where $\frac{y_2 - y_1}{x_2 - x_1}$ is its slope.
- **Point of intersection** of two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is given by
  $$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}\right)$$
- **Angle between two lines :**
  The angle θ between two lines whose slopes are m_1 and m_2 is given by
  $$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2}$$
  If θ is angle between two lines then $\pi - \theta$ is also the angle between them.
- The equation of any straight line parallel to a given line $ax + by + c = 0$ is $ax + by + k = 0$.
- The equation of any straight line perpendicular to a given line, $ax + by + c = 0$ is $bx - ay + k = 0$.
- The equation of any straight line passing through the point of intersection of two given lines $\ell_1 \equiv a_1x + b_1y + c_1 = 0$ and $\ell_2 \equiv a_2x + b_2y + c_2 = 0$ is $\ell_1 + \lambda\ell_2 = 0$
  where λ is any real number, which can be determined by given additional condition in the question.
- The length of perpendicular from a given point (x_1, y_1) to a given line $ax + by + c = 0$ is
  $$\frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}} = p \text{ (say)}$$
  In particular, the length of perpendicular from origin (0, 0) to the line $ax + by + c = 0$ is $\frac{c}{\sqrt{a^2 + b^2}}$
- **Equation of Bisectors :**
  The equations of the bisectors of the angles between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are
  $$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$
- **Distance between parallel lines :**
  Choose a convenient point on any of the lines (put x = 0 and find the value of y or put y = 0 and find the value of x). Now the perpendicular distance from this point on the other line will give the required distance between the given parallel lines.

Pair of straight lines :

- The equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines passing through the origin.

- Let the lines represented by $ax^2 + 2hxy + by^2 = 0$ be $y - m_1x = 0$ and $y - m_2x = 0$, then

  $$m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1m_2 = \frac{a}{b}$$

- General equation of second degree in x, y is
  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ...(i)
  This equation represents two straight lines, if
  $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

  or $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

  and point of intersection of these lines is given by $\left(\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2}\right)$

- The angle between the two straight lines represented by (i) is given by

  $$\tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a + b}$$

- If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel straight lines, then the distance between them is given by

  $$2\sqrt{\frac{g^2 - ac}{a(a + b)}} \text{ or } 2\sqrt{\frac{f^2 - bc}{b(a + b)}}$$

Circle:

Different forms of the equations of a circle :

- **Centre radius form :** the equation of a circle whose centre is the point (h, k) and radius 'a' is
  $(x - h)^2 + (y - k)^2 = a^2$
- **General equation of a circle :** It is given by
  $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(i)
  Equation (i) can also be written as

  $|x - (-g)|^2 + |y - (-f)|^2 = |\sqrt{g^2 + f^2 - c}\,|^2$

  which is in centre-radius form, so by comparing, we get the coordinates of **centre** $(-g, -f)$ and **radius** is $\sqrt{g^2 + f^2 - c}$.
- **Parametric Equations of a Circle :**
  The parametric equations of a circle
  $(x - h)^2 + (y - k)^2 = a^2$ are $x = h + a\cos\theta$ and
  $y = k + a\sin\theta$, where θ is a parameter.
- Lengths of intercepts on the coordinate axes made by the circle (i) are $2\sqrt{g^2 - c}$ and $2\sqrt{f^2 - c}$
- Equation of the circle on the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ as diameter is given by

  $$\left(\frac{y - y_1}{x - x_1}\right)\left(\frac{y - y_2}{x - x_2}\right) = 1$$

- If C_1, C_2 are the centres and a_1, a_2 are the radii of two circles, then
  (i) The circles touch each other externally, if
  $C_1C_2 = a_1 + a_2$
  (ii) The circles touch each other internally, if
  $C_1C_2 = |a_1 - a_2|$
  (iii) The circles intersects at two points, if
  $|a_1 - a_2| < C_1C_2 < a_1 + a_2$
  (iv) The circles neither intersect nor touch each other, if
  $C_1C_2 > a_1 + a_2$ or $C_1C_2 < |a_1 - a_2|$
- Equation of any circle through the point of intersection of two given circles $S_1 = 0$ and $S_2 = 0$ is given by $S_1 + \lambda S_2 = 0$ $(\lambda \neq -1)$ and λ can be determined by an additional condition.
- Equation of the tangent to the given circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at any point (x_1, y_1) on it, is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$
- The straight line $y = mx + c$ touches the circle $x^2 + y^2 = a^2$, if $c^2 = a^2(1 + m^2)$ and the point of contact of the tangent $y = mx \pm a\sqrt{1 + m^2}$, is $\left(\frac{\mp ma}{\sqrt{1 + m^2}}, \frac{\pm a}{\sqrt{1 + m^2}}\right)$
- Length of tangent drawn from the point (x_1, y_1) to the circle $S = 0$ is $\sqrt{S_1}$, where
  $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$
- The equation of pair of tangents drawn from point (x_1, y_1) to the circle
  $S = 0$ i.e. $x^2 + y^2 + 2gx + 2fy + c = 0$, is $SS_1 = T^2$, where $T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$ and S_1 as mentioned above.
- **Chord with a given Middle point :**
  the equation of the chord of the circle $S = 0$ whose mid-point is (x_1, y_1) is given by $T = S_1$, where T and S_1 as defined a above.
- If θ be the angle at which two circles of radii r_1 and r_2 intersect, then

  $$\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$$

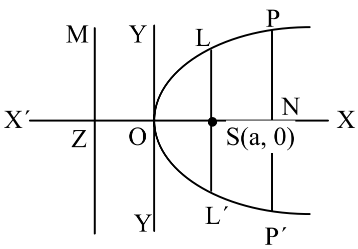
  where d is distance between their centres.

  Note — Two circles are said to be intersect **orthogonally** if the angle between their tangents at their point of intersection is a right angle i.e.
  $r_1^2 + r_2^2 = d^2$ or
  $2g_1g_2 + 2f_1f_2 = c_1 + c_2$
- **Radical axis :** The equation of the radical axis of the two circle is $S_1 - S_2 = 0$ i.e.
  $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$

9-CONIC SECTION

Parabola :

The locus of a point which moves such that its distance from a fixed point is equal to its distance from a fixed straight line, i.e. e = 1 is called a parabola.



Its equation in standard form is $y^2 = 4ax$

(i) Focus S (a, 0)

(ii) Equation of directrix ZM is x + a = 0

(iii) Vertex is O (0, 0)

(iv) Axis of parabola is X′OX

Some definitions :

Focal distance : The distance of a point on parabola from focus is called focal distance. If $P(x_1, y_1)$ is on the parabola, then focal distance is $x_1 + a$.

Focal chord : The chord of parabola which passes through focus is called focal chord of parabola.

Latus rectum : The chord of parabola which passes through focus and perpendicular to axis of parabola is called latus rectum of parabola. Its length is 4a and end points are L(a, 2a) and L′(a, – 2a).

Double ordinate : Any chord which is perpendicular to the axis of the parabola is called its double ordinate.

- **Equation of tangent at** $P(x_1, y_1)$ is

  $yy_1 = 2a(x + x_1)$

  and equation of tangent in slope form is

  $$y = mx + \frac{a}{m}$$

  Here point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

- **Equation of normal at** $P(x_1, y_1)$ is

  $$y - y_1 = \frac{-y_1}{2a}(x - x_1)$$

  and equation of normal in slope form is

  $y = mx - 2am - am^3$

  Here foot of normal is $(am^2, -2am)$

- The line y = mx + c may be tangent to the parabola if c = a/m and may be normal to the parabola if $c = -2am - am^3$.
- Chord of contact at point (x_1, y_1) is

  $yy_1 = 2a(x + x_1)$

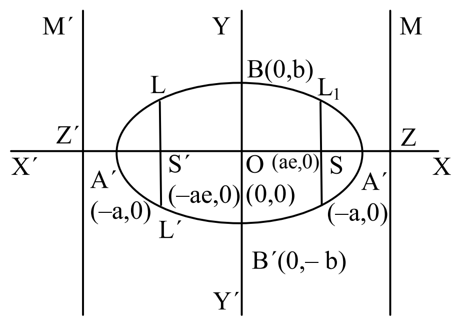
Ellipse :

If a point moves in a plane in such a way that ratio of its distances from a fixed point (focus) and a fixed straight line (directrix) is always less than 1, i.e. e < 1 called an **ellipse**

- Standard equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

  where $b^2 = a^2(1 - e^2)$

  Now, When a > b



In this position,

(i) Major axis 2a and minor axis 2b

(ii) Foci, S′(–ae, 0) and S(ae, 0) and centre O(0, 0)

(iii) Vertices A′ (–a, 0) and A(a, 0)

(iv) Equation of directries ZM and Z′M′ are

$$x \pm \frac{a}{e} = 0, Z\left(\frac{a}{e}, 0\right) \text{ and } Z'\left(-\frac{a}{e}, 0\right)$$

(v) Length of latus rectum is $\frac{2b^2}{a} = LL' = L_1L_1'$

- The coordinates of points of intersection of line y = mx + c and the ellipse are given by

$$\left(\frac{-a^2m}{\sqrt{b^2 + a^2m^2}}, \frac{b^2}{\sqrt{b^2 + a^2m^2}}\right)$$

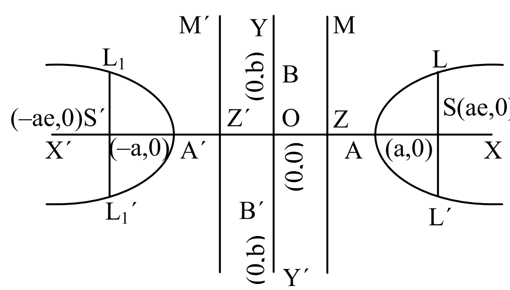
- Equation of tangents of ellipse in term of m is $y = mx \pm \sqrt{b^2 + a^2m^2}$
  and the line y = mx + c is a tangent of the ellipse, if $c = \pm \sqrt{b^2 + a^2m^2}$
- The length of chord cuts off by the ellipse from the line y = mx + c is
  $$\frac{2ab\sqrt{1+m^2}.\sqrt{a^2m^2+b^2-c^2}}{b^2+a^2m^2}$$
- The equation of tangent at any point (x_1, y_1) on the ellipse is
  $$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$
  and at the point (a cos ϕ, b sin ϕ) on the ellipse, the tangents is
  $$\frac{x\cos\phi}{a} + \frac{y\sin\phi}{b} = 1$$
  Parametric equations of the ellipse are
  x = a cos θ and y = b sin θ.
- The equation of normal at any point (x_1, y_1) on the ellipse is
  $$\frac{(x-x_1)a^2}{x_1} = \frac{(y-y_1)b^2}{y_1}$$
  also at the point (a cos ϕ, b sin ϕ) on the ellipse, the equation of normal is
  $$ax \sec\phi - by \operatorname{cosec}\phi = a^2 - b^2$$
- Focal distance of a point $P(x_1, y_1)$ are $a \pm ex_1$
- Chord of contact at point (x_1, y_1) is
  $$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$
- Chord whose mid-point is (h, k) is
  $$\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2} \text{ i.e. } T = S_1$$
- The locus of point of intersection of two perpendicular tangents drawn on the ellipse is $x^2 + y^2 = a^2 + b^2$. This locus is a circle whose centre is the centre of the ellipse and radius is length of line joining the vertices of major and minor axis. This circle is called "director circle".
- The eccentric angle of point P on the ellipse is made by the major axis with the line PO, where O is centre of the ellipse.
- (a) The sum of the focal distance of any point on an ellipse is equal to the major axis of the ellipse.
  (b) The point (x_1, y_1) lies outside, on or inside the ellipse f(x, y) = 0 according as $f(x_1, y_1) > = \text{or} < 0$.
- The locus of mid-point of parallel chords of an ellipse is called its **diameter** and its equation is $y = \frac{-b^2x}{a^2m}$ which is passes through centre of the ellipse.
- The two diameter of an ellipse each of which bisect the parallel chords of others are called **conjugate diameters.** Therefore, the two diameters $y = m_1x$ and $y = m_2x$ will be conjugate diameter if $m_1m_2 = -\frac{b^2}{a^2}$.

Hyperbola :

When the ratio (defined in parabola and ellipse) is greater than 1, i.e. e > 1, then the conic is said to be hyperbola.

Since the equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ differs from that of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in having $-b^2$, most of the results proved for the ellipse are true for the hyperbola, if we replace b^2 by $-b^2$ in their proofs. We therefore, give below the list of corresponding results applicable in case of hyperbola.

- Standard equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $b^2 = a^2(e^2 - 1)$



In this case,

- Foci are S (ae, 0) and S′(–ae, 0).
- Equation of directrices ZM and Z′M′ are
  $x \mp \frac{a}{e} = 0$, $Z\left(\frac{a}{e}, 0\right)$ and $Z'\left(-\frac{a}{e}, 0\right)$
- Transverse axis AA′ = 2a, conjugate axis BB′ = 2b.
- Centre O (0, 0).
- Length of latus rectum $LL' = L_1L_1' = \frac{2b^2}{a}$
- The difference of focal distance from any point $P(x_1, y_1)$ on hyperbola remains constant and is equal to the length of transverse axis. i.e.
  $S'P \sim SP = (ex_1 + a) - (ex_1 - a) = 2a$
- The equation of **rectangular hyperbola**
  $x^2 - y^2 = a^2 = b^2$ i.e. in standard form of hyperbola put a = b. Hence $e = \sqrt{2}$ for rectangular hyperbola.

10-LIMITS,CONTINUITY AND DIFFERENTIABILITY

Limits :

Theorems of Limits :

If f(x) and g(x) are two functions, then

(i) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

(ii) $\lim_{x\to a}[f(x).g(x)] = \lim_{x\to a} f(x) \,.\, \lim_{x\to a} g(x)$

(iii) $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

(iv) $\lim_{x\to a}[kf(x)] = k \lim_{x\to a} f(x)$, where k is constant.

(v) $\lim_{x\to a} \sqrt{f(x)} = \sqrt{\lim_{x\to a} f(x)}$

(vi) $\lim_{x\to a} |f(x)|^{p/q} = \left(\lim_{x\to a} f(x)\right)^{p/q}$, where p and q are integers.

Some important expansions :

(i) $\sin x = \left\{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...\right\}$

(ii) $\cos x = \left\{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...\right\}$

(iii) $\sin h\, x = \left\{x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...\infty\right\}$

(iv) $\cos h\, x = \left\{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...\infty\right\}$

(v) $\tan x = \left\{x + \frac{x^3}{3} + \frac{2x^5}{15} + ...\right\}$

(vi) $\log(1 + x) = \left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\right\}$

(vii) $e^x = \left\{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right\}$

(viii) $a^x = \left\{1 + x\log a + \frac{x^2}{2!}(\log a)^2 + ...\right\}$

(ix) $(1 - x)^{-1} = \{1 + x + x^2 + x^3 + ...\}$

(x) $\sin^{-1}x = \left\{x + \frac{1}{2}.\frac{x^3}{3} + \frac{1}{2}.\frac{3}{4}.\frac{x^5}{5} + \frac{1}{2}.\frac{3}{4}.\frac{5}{6}.\frac{x^7}{7} + ...\right\}$

(xi) $\tan^{-1}x = \left\{x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - .....\right\}$

Some important Limits :

(i) $\lim_{x\to 0} \sin x = 0$

(ii) $\lim_{x\to 0} \cos x = 1$

(iii) $\lim_{x\to 0} \frac{\sin x}{x} = 1 = \lim_{x\to 0} \frac{x}{\sin x}$

(iv) $\lim_{x\to 0} \frac{\tan x}{x} = 1 = \lim_{x\to 0} \frac{x}{\tan x}$

(v) $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

(vi) $\lim_{x\to 0} e^x = 1$

(vii) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

(viii) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$

(ix) $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

(x) $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e = \lim_{x\to -\infty} \left(1 + \frac{1}{x}\right)^x$

(xi) $\lim_{x\to 0} (1 + x)^{1/x} = e$

(xii) $\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = e^a$

(xiii) $\lim_{x\to\infty} a^n = \begin{cases} \infty, & \text{if } a > 1 \\ 0, & \text{if } a < 1 \end{cases}$

i.e. $a^\infty = \infty$, if $a > 1$ and $a^\infty = 0$, if $a < 1$

(xiv) $\lim_{x\to 0} \frac{(1+x)^n - 1}{x} = n$

(xv) $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1 = \lim_{x\to 0} \frac{\tan^{-1} x}{x}$

(xvi) $\lim_{x\to a} \sin^{-1} x = \sin^{-1}a, |a| \le 1$

(xvii) $\lim_{x\to a} \cos^{-1} x = \cos^{-1}a, |a| \le 1$

(xviii) $\lim_{x\to a} \tan^{-1} x = \tan^{-1}a, -\infty < a < \infty$

(xix) $\lim_{x\to e} \log_e x = 1$

(xx) $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

Let $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = m$, then

(xxi) $\lim_{x\to a} (f(x))^{g(x)} = \ell^m$

(xxii) If f(x) ≤ g(x) for every x in the deleted neighbourhood (nbd) of a, then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$.

(xxiii) If f(x) ≤ g(x) ≤ h(x) for every x in the deleted nbd of a and $\lim_{x\to a} f(x) = \ell = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = \ell$.

(xxiv) $\lim_{x\to a} fog(x) = f\left(\lim_{x\to a} g(x)\right) = f(m)$

In particular (a) $\lim_{x\to a} \log f(x) = \log\left(\lim_{x\to a} f(x)\right) = \log \ell$

(b) $\lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a} f(x)} = e^{\ell}$

(xxv) If $\lim_{x\to a} f(x) = +\infty$ or $-\infty$, then $\lim_{x\to a} \frac{1}{f(x)} = 0$.

Evaluation of Limits (Working Rules) :

By factorisation : To evaluate $\lim_{x\to a} \frac{\phi(x)}{\psi(x)}$, factorise both φ(x) and ψ(x), if possible, then cancel the common factor involving a from the numerator and the denominator. In the last obtain the limit by substituting a for x.

Evaluation by substitution : To evaluate $\lim_{x\to a} f(x)$, put x = a + h and simplify the numerator and denominator, then cancel the common factor involving h in the numerator and denominator. In the last obtain the limit by substituting h = 0.

By L – Hospital's rule : Apply L-Hospital's rule to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f^n(x)}{g^n(x)}$$

By using expansion formulae : The expansion formulae can also be used with advantage in simplification and evaluation of limits.

By rationalisation : In case if numerator or denominator (or both) are irrational functions, rationalisation of numerator or denominator (or both) helps to obtain the limit of the function.

Continuity :

f(x) is continuous at x = a if $\lim_{x\to a} f(x)$ exists and is equal to f(a) i.e. if $\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x)$.

Discontinuous functions : A function f is said to be discontinuous at a point a of its domain D if is not continuous there at. The point a is then called a point of discontinuity of the function. The discontinuity may arise due to any of the following situations:

(a) $\lim_{x\to a+} f(x)$ or $\lim_{x\to a-} f(x)$ of both may not exist.

(b) $\lim_{x\to a+} f(x)$ as well as $\lim_{x\to a-} f(x)$ may exist but are unequal.

(c) $\lim_{x\to a+} f(x)$ as well as $\lim_{x\to a-} f(x)$ both may exist but either of the two or both may not be equal to f(a).

We classify the point of discontinuity according to various situations discussed above.

Removable discontinuity : A function f is said to have removable discontinuity at x = a if

$\lim_{x\to a-} f(x) = \lim_{x\to a+} f(x)$ but their common value is not equal to f(a). Such a discontinuity can be removed by assigning a suitable value to the function f at x = a.

Discontinuity of the first kind : A function f is said to have a discontinuity of the first kind at x = a if $\lim_{x\to a-} f(x)$ and $\lim_{x\to a+} f(x)$ both exist but are not equal.

f is said to have a discontinuity of the first kind from the left at x = a if $\lim_{x\to a-} f(x)$ exists but not equal to f(a). Discontinuity of the first kind from the right is similarly defined.

Discontinuity of second kind : A function f is said to have a discontinuity of the second kind at x = a if neither $\lim_{x\to a-} f(x)$ nor $\lim_{x\to a+} f(x)$ exists.

f if said to have discontinuity of the second kind from the left at x = a if $\lim_{x\to a-} f(x)$ does not exist.

Similarly, if $\lim_{x\to a+} f(x)$ does not exist, then f is said to have discontinuity of the second kind from the right at x = a.

Differentiability :

f(x) is said to be differentiable at x = a if R′ = L′

i.e. $\mathop{Lt}_{h\to 0} \frac{f(a+h)-f(a)}{h} = \mathop{Lt}_{h\to 0} \frac{f(a-h)-f(a)}{-h}$

Note : We discuss R, L or R′, L′ at x = a when the function is defined differently for x > a or x < a and at x = a.

11-FUNCTIONS

Definition of a Function :

Let A and B be two sets and f be a rule under which every element of A is associated to a unique element of B. Then such a rule f is called a function from A to B and symbolically it is expressed as

$$f : A \to B$$

or $$A \xrightarrow{f} B$$

Function as a Set of Ordered Pairs

Every function $f : A \to B$ can be considered as a set of ordered pairs in which first element is an element of A and second is the image of the first element. Thus

$$f = \{a, f(a) / a \in A, f(a) \in B\}.$$

Domain, Codomain and Range of a Function :

If $f : A \to B$ is a function, then A is called domain of f and B is called codomain of f. Also the set of all images of elements of A is called the range of f and it is expressed by f(A). Thus

$$f(A) = \{f(a) | a \in A\}$$

obviously $f(A) \subset B$.

Note : Generally we denote domain of a function f by D_f and its range by R_f.

Equal Functions :

Two functions f and g are said to be equal functions if

- domain of f = domain of g
- codomain of f = codomain of g
- $f(x) = g(x)\ \forall\ x$.

Algebra of Functions :

If f and g are two functions then their sum, difference, product, quotient and composite are denoted by

$f + g,\ f - g,\ fg,\ f/g,\ fog$

and they are defined as follows :

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x) f(g)$$

$$(f/g)(x) = f(x)/g(x) \quad (g(x) \neq 0)$$

$$(fog)(x) = f[g(x)]$$

Formulae for domain of functions :

- $D_{f \pm g} = D_f \cap D_g$
- $D_{fg} = D_f \cap D_g$
- $D_{f/g} = D_f \cap D_g \cap \{x | g(x) \neq 0\}$
- $D_{gof} = \{x \in D_f \mid f(x) \in D_g\}$
- $D_{\sqrt{f}} = D_f \cap \{x | f(x) \geq 0\}$

Classification of Functions

1. Algebraic and Transcendental Functions :

- **Algebraic functions :** If the rule of the function consists of sum, difference, product, power or roots of a variable, then it is called an algebraic function.
- **Transcendental Functions :** Those functions which are not algebraic are named as transcendental or non algebraic functions.

2. Even and Odd Functions :

- **Even functions :** If by replacing x by –x in f(x) there in no change in the rule then f(x) is called an even function. Thus
  f(x) is even $\Leftrightarrow f(-x) = f(x)$

- **Odd function :** If by replacing x by –x in f(x) there is only change of sign of f(x) then f(x) is called an odd function. Thus

  f(x) is odd $\Leftrightarrow$ f(–x) = – f(x)

3. Explicit and Implicit Functions :

- **Explicit function :** A function is said to be explicit if its rule is directly expressed (or can be expressed(in terms of the independent variable. Such a function is generally written as

  y = f(x), x = g(y) etc.

- **Implicit function :** A function is said to be implicit if its rule cannot be expressed directly in terms of the independent variable. Symbolically we write such a function as

  f(x, y) = 0, ϕ(x, y) = 0 etc.

4. Continuous and Discontinuous Functions :

- **Continuous functions :** A functions is said to be continuous if its graph is continuous i.e. there is no gap or break or jump in the graph.

- **Discontinuous Functions** : A function is said to be discontinuous if it has a gap or break in its graph atleast at one point. Thus a function which is not continuous is named as discontinuous.

5. Increasing and Decreasing Functions :

- **Increasing Functions :** A function f(x) is said to be increasing function if for any x_1, x_2 of its domain

  $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$

  or $x_1 > x_2 \Rightarrow f(x_1) \ge f(x_2)$

- **Decreasing Functions :** A function f(x) is said to be decreasing function if for any x_1, x_2 of its domain

  $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$

  or $x_1 > x_2 \Rightarrow f(x_1) \le f(x_2)$

Periodic Functions :

A functions f(x) is called a periodic function if there exists a positive real number T such that

$$f(x + T) = f(x). \qquad \forall x$$

Also then the least value of T is called the period of the function f(x).

Period of f(x) = T

$\Rightarrow$ Period of f(nx + a) = T/n

Periods of some functions :

Function	Period
sin x, cos x, sec x, cosec x,	2π
tan x, cot x	π
$\sin^n x$, $\cos^n x$, $\sec^n x$, $\text{cosec}^n x$	2π if n is odd π if n is even
$\tan^n x$, $\cot^n x$	$\pi \ \forall \ n \in N$
|sin x|, |cos x|, |sec x|, |cosec x|	π
|tan x|, |cot x|,	π
|sin x| + |cos x|, $\sin^4 x + \cos^4 x$ |sec x| + |cosec x|	$\frac{\pi}{2}$
|tan x| + |cot x|	$\frac{\pi}{2}$
x – [x]	1
• Period of f(x) = T $\Rightarrow$ period of f(ax + b) = T/|a|	
• Period of $f_1(x) = T_1$, period fo $f_2(x) = T_2$ $\Rightarrow$ period of a $f_1(x)$ + b$f_2(x)$ $\le$ LCM {T_1, T_2}	

Kinds of Functions :

- **One-one/ May one Functions :**

  A function f : A $\to$ B is said to be one-one if different elements of A have their different images in B.

  Thus

$$f \text{ is one-one} \Leftrightarrow \begin{cases} a \ne b \quad \Rightarrow \quad f(a) \ne f(b) \\ \qquad\qquad \text{or} \\ f(a) = f(b) \quad \Rightarrow \quad a = b \end{cases}$$

  A function which is not one-one is called many one. Thus if f is many one then atleast two different elements have same f-image.

- **Onto/Into Functions :** A function f : A $\to$ B is said to be onto if range of f = codomain of f

  Thus f is onto $\Leftrightarrow$ f(A) = B

Hence f : A → B is onto if every element of B (co-domain) has its f–preimage in A (domain).

A function which is not onto is named as into function. Thus f : A → B is into if f(A) ≠ B. i.e., if there exists atleast one element in codomain of f which has no preimage in domain.

Note :

Total number of functions : If A and B are finite sets containing m and n elements respectively, then

- total number of functions which can be defined from A to B = n^m.
- total number of one-one functions from A to B

$$= \begin{cases} {}^n p_m & \text{if} \quad m \le n \\ 0 & \text{if} \quad m > n \end{cases}$$

- total number of onto functions from A to B (if m ≥ n) = total number of different n groups of m elements.

Composite of Functions :

Let f : A → B and g : B → C be two functions, then the composite of the functions f and g denoted by gof, is a function from A to C given by gof : A → C, (gof) (x) = g[f(x)].

Properties of Composite Function :

The following properties of composite functions can easily be established.

- Composite of functions is not commutative i.e.,

$$fog \ne gof$$

- Composite of functions is associative i.e.

$$(fog)oh = fo(goh)$$

- Composite of two bijections is also a bijection.

Inverse Function :

If f : A → B is one-one onto, then the inverse of f i.e., f^{-1} is a function from B to A under which every b ∈ B is associated to that a ∈ A for which f(a) = b.

Thus $\quad f^{-1} : B \to A,$

$$f^{-1}(b) = a \Leftrightarrow f(a) = b.$$

Domain and Range of some standard functions :

Function	Domain	Range
Polynomial function	R	R
Identity function x	R	R
Constant function c	R	{c}
Reciprocal function 1/x	R_0	R_0
x^2, |x|	R	$R^+ \cup \{0\}$
x^3, x |x|	R	R
Signum function	R	{–1, 0, 1}
x + |x|	R	$R^+ \cup \{0\}$
x – |x|	R	$R^- \cup \{0\}$
[x]	R	Z
x – [x]	R	[0, 1)
$\sqrt{x}$	[0, ∞)	[0, ∞)
a^x	R	R^+
log x	R^+	R
sin x	R	[–1, 1]
cos x	R	[–1, 7]
tan x	R – {± π/2, ± 3π/2, ...}	R
cot x	R – {0, ± π. ± 2π, ...	R
sec x	R – (± π/2, ± 3π/2, ...	R – (–1, 1)
cosec x	R – {0, ±π, ± 2π, ...}	R –(–1, 1)
sinh x	R	R
cosh x	R	[1, ∞)
tanh x	R	(–1, 1)
coth x	R_0	R –[1, –1]
sech x	R	(0, 1]
cosech x	R_0	R_0
$\sin^{-1}$ x	[–1, 1]	[–π/2, π/2]
$\cos^{-1}$x	[–1, 1]	[0, π]
$\tan^{-1}$x	R	(–π/2, π/2}
$\cot^{-1}$ x	R	(0, π)
$\sec^{-1}$ x	R –(–1, 1)	[0, π] – {π/2}
cosec^{-1}x	R – (–1, 1)	(– π/2, π/2] – {0}

12-DIFFERENTIATION

Differentiation and Applications of Derivatives :

- If y = f(x), then

  1. $\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

  2. $\left(\frac{dy}{dx}\right)_{x=a} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

  3. $\left(\frac{dy}{dx}\right)_{x=a} = \lim_{x\to h}\frac{f(a+h)-f(a)}{h}$

- If u = f(x), v = ϕ(x), then

  1. $\frac{d}{dx}(k) = 0$

  2. $\frac{d}{dx}(ku) = k\frac{du}{dx}$

  3. $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

  4. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

  5. $\frac{du}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

  6. If x = f(t), y = ϕ (t), then $\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt}$

  7. If y = f[ϕ(x)], then $\frac{dy}{dx} = f'[\phi(x)].\frac{d}{dx}[\phi(x)]$

  8. If w = f(y), then $\frac{dw}{dx} = f'(y)\frac{dy}{dx}$

  9. If y = f(x), z = ϕ(x), then $\frac{dy}{dz} = \frac{dy}{dx}.\frac{dx}{dz}$

  10. $\frac{dy}{dx}.\frac{dx}{dy} = 1$ or $\frac{dy}{dx} = \frac{1}{dx/dy}$

- 1. $\frac{d}{dx}(k) = 0$

  2. $\frac{d}{dx}x^n = nx^{n-1}$

  3. $\frac{d}{dx}\frac{1}{x^n} = -\frac{n}{x^{n+1}}$

  4. $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

  5. $\frac{d}{dx}e^x = e^x$

  6. $\frac{d}{dx}a^x = a^x \log a$

  7. $\frac{d}{dx}\log x = \frac{1}{x}$

  8. $\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e$

  9. $\frac{d}{dx}\sin x = \cos x$

  10. $\frac{d}{dx}\cos x = -\sin x$

  11. $\frac{d}{dx}\tan x = \sec^2 x$

  12. $\frac{d}{dx}\cot x = -\text{cosec}^2 x$

  13. $\frac{d}{dx}\sec x = \sec x \tan x$

  14. $\frac{d}{dx}\text{cosec}\, x = -\text{cosec}\, x \cot x$

  15. $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

  16. $\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$

  17. $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

  18. $\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$

  19. $\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$

  20. $\frac{d}{dx}\text{cosec}^{-1}x = -\frac{1}{x\sqrt{x^2-1}}$

- **Suitable substitutions :** The functions any also be reduced to simplar forms by the substitutions as follows.
  1. If the function involve the term $\sqrt{(a^2 - x^2)}$, then put x = a sin θ or x = a cos θ.
  2. If the function involve the term $\sqrt{(a^2 + x^2)}$, then put x = a tan θ or x = a cot θ.
  3. If the function involve the term $\sqrt{(x^2 - a^2)}$, then put x = a sec θ or x = a cosec θ.
  4. If the function involve the term $\sqrt{\frac{a-x}{a+x}}$, then put x = a cos θ or x = a cos 2θ

  All the above substitutions are also true, if a = 1
- **Differentiation by taking logarithm :**

  Differentiation of the functions of the following types are obtained by taking logarithm.
  1. When the functions consists of the product and quotient of a number of functions.
  2. When a function of x is raised to a power which is itself a function of x.

  For example, let $y = [f(x)]^{\phi(x)}$

  Taking logarithm of both sides, log y = φ(x) log f(x)

  Differentiating both sides w.r.t 'x',

  $$\frac{1}{y}\frac{dy}{dx} = \phi'(x)\log f(x) + \phi(x).\frac{f'(x)}{f(x)}$$

  $$= [f(x)]^{\phi(x)} \log f(x).\phi'(x) + \phi(x).[f(x)]^{\phi(x)-1}.f'(x)$$

  $\frac{dy}{dx}$ = Differential of y treading f(x) as constant + Differential of y treating φ(x) as constant.

  It is an important formula.
- **Differentiation of implicit functions :**
  1. If f(x, y) = 0 is a implicit function, then

  $$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

  $$= -\frac{\text{Diff. of f w.r.t. x keeping y constant}}{\text{Diff. of f w.r.t. y keeping x constant}}$$

  For example, consider $f(x, y) = x^2 + 3xy + y^2 = 0$, then

  $$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{2x+3y}{3x+2y}$$

  1. If y = f(x), then

  $$\frac{dy}{dx} = y_1 = f'(x),\quad \frac{d^2y}{dx^2} = y_2 = f''(x), ...$$

  $$\frac{d^2y}{dx^n} = y_n = f^n(x)$$

  2. $\frac{d^n}{dx^n}(ax+b)^n = n!\,a^n$
  3. $\frac{d^n}{dx^n}(ax+b)^m = m(m-1) .... (m-n+1)\,a^n(ax+b)^{m-n}$
  4. $\frac{d^n}{dx^n}e^{mx} = m^n e^{mx}$
  5. $\frac{d^n}{dx^n}a^{mx} = m^n a^{mx}(\log a)^n$
  6. $\frac{d^n}{dx^n}\log(ax+b) = \frac{(-1)^{n-1}a^n(n-1)!}{(ax+b)^n}$
  7. $\frac{d^n}{dx^n}\sin(ax+b) = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$
  8. $\frac{d^n}{dx^n}\cos(ax+b) = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$
- **Leibnitz's theorem :** If u and v are any two functions of x such that their desired differential coefficients exist, then the n^{th} differential coefficient of uv is given by

  $$D^n(uv) = (D^n u)v + {}^nC_1(D^{n-1}u)(Dv) + {}^nC_2(D^{n-2}u)(D^2v) + ... + u(D^n v)$$

Do you know

- Did you know that there are 206 bones in the adult human body and there are 300 in children (as they grow some of the bones fuse together).
- Flea's can jump 130 times higher than their own height. In human terms this is equal to a 6ft. person jumping 780 ft. into the air.
- The most dangerous animal in the world is the common housefly. Because of their habits of visiting animal waste, they transmit more diseases than any other animal.
- Snakes are true carnivorous because they eat nothing but other animals. They do not eat any type of plant material.
- The world's largest amphibian is the giant salamander. It can grow up to 5 ft. in length.
- The smallest bone in the human body is the stapes or stirrup bone located in the middle ear. It is approximately .11 inches (.28 cm) long.

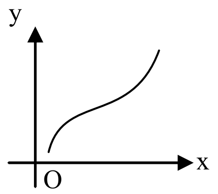
13-MONOTONICITY,MAXIMA AND MINIMA

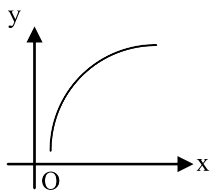
Monotonic Functions :

A function f(x) defined in a domain D is said to be

(i) **Monotonic increasing :**

$$\Leftrightarrow \begin{cases} x_1 < x_2 \Rightarrow f(x_1) \le f(x_2) \\ x_1 > x_2 \Rightarrow f(x_1) \ge f(x_2) \end{cases} \forall\ x_1, x_2 \in D$$

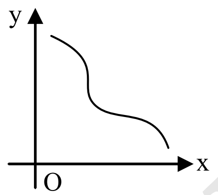


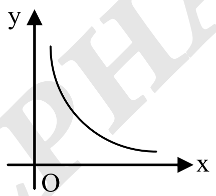


$$\text{i.e.,} \Leftrightarrow \begin{cases} x_1 < x_2 \Rightarrow f(x_1) \ngtr f(x_2) \\ x_1 > x_2 \Rightarrow f(x_1) \nless f(x_2) \end{cases} \forall\ x_1, x_2 \in D$$

(ii) **Monotonic decreasing :**

$$\Leftrightarrow \begin{cases} x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2) \\ x_1 > x_2 \Rightarrow f(x_1) \le f(x_2) \end{cases} \forall\ x_1, x_2 \in D$$





$$\text{i.e.,} \Leftrightarrow \begin{cases} x_1 < x_2 \Rightarrow f(x_1) \nless f(x_2) \\ x_1 > x_2 \Rightarrow f(x_1) \ngtr f(x_2) \end{cases} \forall\ x_1, x_2 \in D$$

A function is said to be monotonic function in a domain if it is either monotonic increasing or monotonic decreasing in that domain.

Note : If $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)\ \forall\ x_1, x_2 \in D$, then f(x) is called strictly increasing in domain D and similarly decreasing in D.

Method of testing monotonicity :

(i) **At a point :** A function f(x) is said to be monotonic increasing (decreasing) at a point x = a of its domain if it is monotonic increasing (decreasing) in the interval (a – h, a + h) where h is a small positive number. Hence we may observer that if f(x) is monotonic increasing at x = a then at this point tangent to its graph will make an acute angle with x-axis where as if the function is monotonic decreasing there then tangent will make an obtuse angle with x-axis. Consequently f′(a) will be positive or negative according as f(x) is monotonic increasing or decreasing at x = a.

So at x = a, function f(x) is

monotonic increasing ⇔ f′(a) > 0

monotonic decreasing ⇔ f′(a) < 0

(ii) **In an interval :** In [a, b], f(x) is

$$\left.\begin{matrix} \text{monotonic increasing} \Leftrightarrow f'(x) \ge 0 \\ \text{monotonic decreasing} \Leftrightarrow f'(x) \le 0 \\ \text{constant} \Leftrightarrow f'(x) = 0 \end{matrix}\right\} \forall\ x \in (a, b)$$

Note :

(i) In above results f′(x) should not be zero for all values of x, otherwise f(x) will be a constant function.

(ii) If in [a, b], f′(x) < 0 at least for one value of x and f′(x) > 0 for at least one value of x, then f(x) will not be monotonic in [a, b].

Examples of monotonic function :

If a functions is monotonic increasing (decreasing) at every point of its domain, then it is said to be monotonic increasing (decreasing) function.

In the following table we have example of some monotonic/not monotonic functions

Monotonic increasing	Monotonic decreasing	Not monotonic
x^3	1/x, x > 0	x^2
x|x|	1 – 2x	|x|
e^x	e^{-x}	$e^x + e^{-x}$
log x	$\log_2 x$	sin x
sin h x	cosec h x, x > 0	cos h x
[x]	cot hx, x > 0	sec h x

Properties of monotonic functions :

- If f(x) is strictly increasing in some interval, then in that interval, f^{-1} exists and that is also strictly increasing function.
- If f(x) is continuous in [a, b] and differentiable in (a, b), then

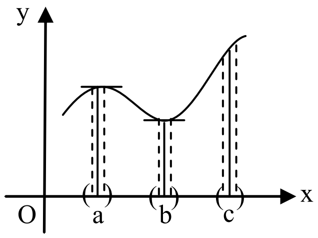
  $f'(c) \ge 0\ \forall\ c \in (a, b) \Rightarrow$ f(x) is monotonic increasing in [a, b]

  $f'(c) \ge 0\ \forall\ c \in (a, b) \Rightarrow$ f(x) is monotonic decreasing in [a, b]

- If both f(x) and g(x) are increasing (or decreasing) in [a, b] and gof is defined in [a, b], then gof is increasing.
- If f(x) and g(x) are two monotonic functions in [a, b] such that one is increasing and other is decreasing then gof, it is defined, is decreasing function.

Maximum and Minimum Points :

The value of a function f(x) is said to be maximum at x = a if there exists a small positive number δ such that $f(a) > f(x)$



Also then the point x = a is called a maximum point for the function f(x).

Similarly the value of f(x) is said to be minimum at x = b if there exists a small positive number δ such that

$f(b) < f(x)\ \forall\ x \in (b - \delta, b + \delta)$

Also then the point x = b is called a minimum point for f(x)

Hence we find that :

(i) x = a is a maximum point of f(x)

$$\Leftrightarrow \begin{cases} f(a) - f(a+h) > 0 \\ f(a) - f(a-h) > 0 \end{cases}$$

(ii) x = b is a minimum point of f(x)

$$\Leftrightarrow \begin{cases} f(b) - f(b+h) < 0 \\ f(b) - f(b-h) > 0 \end{cases}$$

(iii) x = c is neither a maximum point nor a minimum point

$$\Leftrightarrow \left\{ \begin{matrix} f(c) - f(c+h) \\ \text{and} \\ f(c) - f(c-h) \end{matrix} \right\} \text{ have opposite signs.}$$

Where h is a very small positive number.

Note :

- The maximum and minimum points are also known as extreme points.
- A function may have more than one maximum and minimum points.
- A maximum value of a function f(x) in an interval [a, b] is not necessarily its greatest value in that interval. Similarly a minimum value may not be the least value of the function. A minimum value may be greater than some maximum value for a function.
- The greatest and least values of a function f(x) in an interval [a, b] may be determined as follows :

  Greatest value = max. {f(a), f(b), f(c)}

  Least value = min. {f(a), f(b), f(c)}

  where x = c is a point such that $f'(c) = 0$.
- If a continuous function has only one maximum (minimum) point, then at this point function has its greatest (least) value.
- Monotonic functions do not have extreme points.

Conditions for maxima and minima of a function

Necessary condition : A point x = a is an extreme point of a function f(x) if $f'(a) = 0$, provided $f'(a)$ exists. Thus if $f'(a)$ exists, then

$x = a$ is an extreme point $\Rightarrow f'(a) = 0$ or

$f'(a) \neq 0 \Rightarrow x = a$ is not an extreme point

But its converse is not true i.e.

$f'(a) = 0 \not\Rightarrow x = a$ is an extreme point.

For example if $f(x) = x^3$, then $f'(0) = 0$ but x = 0 is not an extreme point.

Sufficient condition : For a given function f(x), a point x = a is

- a maximum point if $f'(a) = 0$ and $f''(a) < 0$
- a minimum point if $f'(a) = 0$ and $f''(a) > 0$
- not an extreme point if $f'(a) = 0 = f''(a)$ and $f'''(a) \neq 0$.

Note : If $f'(a) = 0$, $f''(a) = 0$, $f'''(a) = 0$ then the sign of $f^{(4)}(a)$ will determine the maximum or minimum point as above.

Working Method :

- Find $f'(x)$ and $f''(x)$.
- Solve $f'(x) = 0$. Let its roots be a, b, c, ...
- Determine the sign of $f''(x)$ at x = a, b, c, ... and decide the nature of the point as mentioned above.

Properties of maxima and minima :

If f(x) is continuous function, then

- Between two equal values of f(x), there lie atleast one maxima or minima.
- Maxima and minima occur alternately. For example if x = –1, 2, 5 are extreme points of a continuous function and if x = –1 is a maximum point then x = 2 will be a minimum point and x = 5 will be a maximum point.
- When x passes a maximum point, the sign of dy/dx changes from + ve to – ve, where as when x passes through a minimum point, the sign of $f'(x)$ changes from –ve to + ve.
- If there is no change in the sign of dy/dx on two sides of a point, then such a point is not an extreme point.
- If f(x) is maximum (minimum) at a point x = a, then 1/f(x), [f(x) ≠ 0] will be minimum (maximum) at that point.
- If f(x) is maximum (minimum) at a point x = a, then for any $\lambda \in R$, $\lambda + f(x)$, log f(x) and for any k > 0, k f(x), $[f(x)]^k$ are also maxmimum (minimum) at that point.

14-INTEGRATION

Integration :

- If $\frac{d}{dx}f(x) = F(x)$, then $\int F(x)\,dx = f(x) + c$, where c is an arbitrary constant called constant of integration.

1. $\int x^n dx = \frac{x^{n+1}}{n+1}\ (n \neq -1)$

2. $\int \frac{1}{x} dx = \log x$

3. $\int e^x dx = e^x$

4. $\int a^x dx = \frac{a^x}{\log_e a}$

5. $\int \sin x\, dx = -\cos x$

6. $\int \cos x\, dx = \sin x$

7. $\int \sec^2 x\, dx = \tan x$

8. $\int \operatorname{cosec}^2 x\, dx = -\cot x$

9. $\int \sec x \tan x\, dx = \sec x$

10. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x$

11. $\int \sec x\, dx = \log(\sec x + \tan x) = \log \tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$

12. $\int \operatorname{cosec} x\, dx = -\log(\operatorname{cosec} x + \cot x) = \log \tan\left(\frac{x}{2}\right)$

13. $\int \tan x\, dx = -\log \cos x$

14. $\int \cot x\, dx = \log \sin x$

15. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} = -\cos^{-1}\frac{x}{a}$

16. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} = -\frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right)$

17. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} = -\frac{1}{a}\operatorname{cosec}^{-1}\left(\frac{x}{a}\right)$

18. $\int \frac{1}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|$, when $x > a$

19. $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|$, when $x < a$

20. $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left\{x+\sqrt{x^2-a^2}\right\} = \cos h^{-1}\left(\frac{x}{a}\right)$

21. $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left\{x+\sqrt{x^2+a^2}\right\} = \sin h^{-1}\left(\frac{x}{a}\right)$

22. $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)$

23. $\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2\log\left\{x+\sqrt{x^2-a^2}\right\}$

24. $\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{1}{2}a^2\log\left\{x+\sqrt{x^2+a^2}\right\}$

25. $\int \frac{f'(x)}{f(x)}dx = \log f(x)$

26. $\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)}$

Integration by Decomposition into Sum :

1. **Trigonometrical transformations :** For the integrations of the trigonometrical products such as $\sin^2 x$, $\cos^2 x$, $\sin^3 x$, $\cos^3 x$, sin ax cos bx, etc., they are expressed as the sum or difference of the sines and cosines of multiples of angles.
2. **Partial fractions :** If the given function is in the form of fractions of two polynomials, then for its integration, decompose it into partial fractions (if possible).

Integration of some special integrals :

(i) $\int \frac{dx}{ax^2+bx+c}$

This may be reduced to one of the forms of the above formulae (16), (18) or (19).

(ii) $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

This can be reduced to one of the forms of the above formulae (15), (20) or (21).

(iii) $\int \sqrt{ax^2+bx+c}\,dx$

This can be reduced to one of the forms of the above formulae (22), (23) or (24).

(iv) $\int \frac{(px+q)dx}{ax^2+bx+c}$, $\int \frac{(px+q)dx}{\sqrt{ax^2+bx+c}}$

For the evaluation of any of these integrals, put $px + q = A$ {differentiation of $(ax^2 + bx + c)$} + B

Find A and B by comparing the coefficients of like powers of x on the two sides.

- **1.** If k is a constant, then

$$\int k\,dx = kx \text{ and } \int k\,f(x)\,dx = k\int f(x)\,dx$$

2. $\int \{f_1(x) \pm f_2(x)\}\,dx = \int f_1(x)\,dx \pm \int f_2(x)\,dx$

Some Proper Substitutions :

1. $\int f(ax+b)\,dx$, $ax + b = t$

2. $\int f(ax^n+b)x^{n-1}dx$, $ax^n + b = t$

3. $\int f\{\phi(x)\}\,\phi'(x)\,dx$, $\phi(x) = t$

4. $\int \frac{f'(x)}{f(x)}dx$, $f(x) = t$

5. $\int \sqrt{a^2-x^2}\,dx$, $x = a \sin\theta$ or $a\cos\theta$

6. $\int \sqrt{a^2+x^2}\,dx$, $x = a\tan\theta$

7. $\int \sqrt{\frac{a^2-x^2}{a^2+x^2}}\,dx$, $x^2 = a^2\cos 2\theta$

8. $\int \sqrt{a \pm x}\,dx$, $a \pm x = t^2$

9. $\int \sqrt{\frac{a-x}{a+x}}\,dx$, $x = a\cos 2\theta$

10. $\int \sqrt{2ax-x^2}\,dx$, $x = a(1-\cos\theta)$

11. $\int \sqrt{x^2-a^2}\,dx$, $x = a\sec\theta$

Substitution for Some irrational Functions :

1. $\int \frac{dx}{(px+q)\sqrt{ax+b}}$, $ax + b = t^2$

2. $\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$, $px + q = \frac{1}{t}$

3. $\int \frac{dx}{(px^2+qx+r)\sqrt{ax+b}}$, $ax + b = t^2$

4. $\int \frac{dx}{(px^2+r)\sqrt{ax^2+c}}$, at first $x = \frac{1}{t}$ and then $a + ct^2 = z^2$

Some Important Integrals :

1. To evaluate $\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$, $\int \sqrt{\left(\frac{x-\alpha}{\beta-x}\right)}\,dx$, $\int \sqrt{(x-\alpha)(\beta-x)}\,dx$. Put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

2. To evaluate $\int \frac{dx}{a+b\cos x}$, $\int \frac{dx}{a+b\sin x}$, $\int \frac{dx}{a+b\cos x+c\sin x}$

Replace $\sin x = \frac{\left(2\tan\frac{x}{2}\right)}{\left(1+\tan^2\frac{x}{2}\right)}$ and $\cos x = \frac{\left(1-\tan^2\frac{x}{2}\right)}{\left(1+\tan^2\frac{x}{2}\right)}$

Then put $\tan\frac{x}{2} = t$.

3. To evaluate $\int \frac{p\cos x+q\sin x}{a+b\cos x+c\sin x}dx$

Put $p\cos x + q\sin x = A(a + b\cos x + c\sin x)$ + B. diff. of $(a + b\cos x + c\sin x) + C$

A, B and C can be calculated by equating the coefficients of cos x, sin x and the constant terms.

4. To evaluate $\int \frac{dx}{a\cos^2 x+2b\sin x\cos x+c\sin^2 x}$, $\int \frac{dx}{a\cos^2 x+b}$, $\int \frac{dx}{a+b\sin^2 x}$

In the above type of questions divide N^r and D^r by $\cos^2 x$. The numerator will become $\sec^2 x$ and in the denominator we will have a quadratic equation in tan x (change $\sec^2 x$ into $1 + \tan^2 x$).

Putting tan x = t the question will reduce to the form $\int \frac{dt}{at^2+bt+c}$

5. **Integration of rational function of the given form**

(i) $\int \frac{x^2+a^2}{x^4+kx^2+a^4}dx$, (ii) $\int \frac{x^2-a^2}{x^4+kx^2+a^4}dx$, where k is a constant, positive, negative or zero.

These integrals can be obtained by dividing numerator and denominator by x^2, then putting $x - \frac{a^2}{x} = t$ and $x + \frac{a^2}{x} = t$ respectively.

Integration of Product of Two Functions :

1. $\int f_1(x)\,f_2(x)\,dx = f_1(x)\int f_2(x)\,dx - \int\left[(f_1'(x)\int f_2(x)dx\right]dx$

Proper choice of the first and second functions : Integration with the help of the above rule is called integration by parts, In the above rule, there are two terms on R.H.S. and in both the terms integral of the second function is involve. Therefore in the product of two functions if one of the two functions is not directly integrable (e.g. log x, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc.) we take it as the first function and the remaining function is taken as the second function. If there is no other function, then unity is taken as the second function. If in the integral both the functions are easily integrable, then the first function is chosen in such a way that the derivative of the function is a simple functions and the function thus obtained under the integral sign is easily integrable than the original function.

2. $\int e^{ax} \sin(bx + c)\, dx$

$$= \frac{e^{ax}}{a^2 + b^2}[a \sin(bx + c) - b \cos(bx + c)]$$

$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin\left[bx + c - \tan^{-1}\frac{b}{a}\right]$$

3. $\int e^{ax} \cos(bx + c)\, dx$

$$= \frac{e^{ax}}{a^2 + b^2}[a \cos(bx + c) + b \sin(bx + c)]$$

$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} \cos\left[bx + c - \tan^{-1}\frac{b}{a}\right]$$

4. $\int e^{kx}\{kf(x) + f'(x)\}\, dx = e^{kx}f(x)$

5. $\int \log_e x = x(\log_e x - 1) = x \log_e\left(\frac{x}{e}\right)$

Integration of Trigonometric Functions :

1. To evaluate the integrals of the form

$I = \int \sin^m x \cos^n x\, dx$, where m and n are rational numbers.

(i) Substitute sin x = t, if n is odd;

(ii) Substitute cos x = t, if m is odd;

(iii) Substitute tan x = t, if m + n is a negative even integer; and

(iv) Substitute cot x = t, if $\frac{1}{2}(m + 1) + \frac{1}{2}(n - 1)$ is an integer.

2. Integrals of the form $\int R(\sin x, \cos x)\, dx$, where R is a rational function of sin x and cos x, are transformed into integrals of a rational function by the substitution $\tan\frac{x}{2} = t$, where $-\pi < x < \pi$. This is the so called universal substitution. Sometimes it is more convenient to make the substitution $\cot\frac{x}{2} = t$ for $0 < x < 2\pi$.

The above substitution enables us to integrate any function of the form R (sin x, cos x). However, in practice, it sometimes leads to extremely complex rational functions. In some cases, the integral can be simplified by –

(i) Substituting sin x = t, if the integral is of the form $\int R(\sin x) \cos x\, dx$.

(ii) Substituting cos x = t, if the integral is of the form $\int R(\cos x) \sin x\, dx$.

(iii) Substituting tan x = t, i.e. $dx = \frac{dt}{1 + t^2}$, if the integral is dependent only on tan x.

Some Useful Integrals :

1. (When a > b) $\int \frac{dx}{a + b\cos x}$

$$= \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left[\sqrt{\frac{a - b}{a + b}} \tan\frac{x}{2}\right] + c$$

2. (When a < b) $\int \frac{dx}{a + b\cos x}$

$$= -\frac{1}{\sqrt{b^2 - a^2}} \log\left|\frac{\sqrt{b - a}\tan\frac{x}{a} - \sqrt{a + b}}{\sqrt{b - a}\tan\frac{x}{a} + \sqrt{a + b}}\right|$$

3. (when a = b) $\int \frac{dx}{a + b\cos x} = \frac{1}{a}\tan\frac{x}{2} + c$

4. (When a > b) $\int \frac{dx}{a + b\sin x}$

$$= \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left\{\frac{a\tan\left(\frac{x}{2}\right) + b}{\sqrt{a^2 - b^2}}\right\} + c$$

5. (When a < b) $\int \frac{dx}{a + b\sin x}$

$$= \frac{1}{\sqrt{b^2 - a^2}} \log\left|\frac{a\tan\left(\frac{x}{2}\right) + b - \sqrt{b^2 - a^2}}{a\tan\left(\frac{x}{2}\right) + b + \sqrt{b^2 - a^2}}\right| + c$$

6. (When a = b) $\int \frac{dx}{a + b\sin x} = \frac{1}{a}[\tan x - \sec x] + c$

15-DEFINITE INTEGRAL AND AREAS

Properties 1 :

- If $\int f(x)\,dx = F(x)$, then

  $\int_a^b f(x)\,dx = F(b) - F(a),\ b \ge a$

  Where F(x) is one of the antiderivatives of the function f(x), i.e. $F'(x) = f(x)\ (a \le x \le b)$.

  Remark : When evaluating integrals with the help of the above formula, the students should keep in mind the condition for its legitimate use. This formula is used to compute the definite integral of a function continuous on the interval [a, b] only when the equality $F'(x) = f(x)$ is fulfilled in the whole interval [a, b], where F(x) is antiderivative of the function f(x). In particular, the antiderivative must be a function continuous on the whole interval [a, b]. A discontinuous function used as an antiderivative will lead to wrong result.

- If $F(x) = \int_a^x f(t)\,dt,\ t \ge a$, then $F'(x) = f(x)$

Properties of Definite Integrals :

If $f(x) \ge 0$ on the interval [a, b], then $\int_a^b f(x)\,dx \ge 0$

- $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$
- $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,\ a < c < b$
- $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

or $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

- $\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_a^b f(x)\,dx & \text{if } f(-x) = f(x) \\ 0 & \text{if } f(-x) = -f(x) \end{cases}$
- $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_a^b f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$
- Every continuous function defined on [a, b] is integrable over [a, b].
- Every monotonic function defined on [a, b] is integrable over [a, b]
- If f(x) is a continuous function defined on [a, b], then there exists $c \in (a, b)$such that

  $\int_a^b f(x)\,dx = f(c)\,.\,(b-a)$

  The number $f(c) = \dfrac{1}{(b-a)}\int_a^b f(x)\,dx$ is called the mean value of the function f(x) on the interval [a, b].
- If f is continous on [a, b], then the integral function g defined by $g(x) = \int_a^x f(t)\,dt$ for $x \in [a, b]$ is derivable on [a, b] and $g'(x) = f(x)$ for all $x \in [a, b]$.
- If m and M are the smallest and greatest values of a function f(x) on an interval [a, b], then

  $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$
- If the function $\phi(x)$ and $\psi(x)$ are defined on [a, b] and differentiable at a point $x \in (a, b)$ and f(t) is continuous for $\phi(a) \le t \le \psi(b)$, then

  $\dfrac{d}{dx}\left(\int_{\phi(x)}^{\psi(x)} f(t)\,dt\right) = f(\psi(x))\,\psi'(x) - f(\phi(x))\,\phi'(x)$
- $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$
- If $f^2(x)$ and $g^2(x)$ are integrable on [a, b], then

  $\left|\int_a^b f(x)g(x)\,dx\right| \le \left(\int_a^b f^2(x)\,dx\right)^{1/2}\left(\int_a^b g^2(x)\,dx\right)^{1/2}$
- **Change of variables :** If the function f(x) is continuous on [a, b] and the function $x = \phi(t)$ is continuously differentiable on the interval $[t_1, t_2]$ and $a = \phi(t_1),\ b = \phi(t_2)$, then

  $\int_a^b f(x)\,dx = \int_{t_1}^{t_2} f(\phi(t))\,\phi'(t)\,dt$
- Let a function $f(x, \alpha)$ be continuous for $a \le x \le b$ and $c \le \alpha \le d$. Then for any $\alpha \in [c, d]$, if

  $I(\alpha) = \int_a^b f(x,\alpha)\,dx$, then $I'(\alpha) = \int_a^b f'(x,\alpha)\,dx$, Where $I'(\alpha)$ is the derivative of $I(\alpha)$ w.r.t. α and $f'(x, \alpha)$ is the derivative of $f(x, \alpha)$ w.r.t. α, keeping x constant.

Integrals with Infinite Limits :

If a function f(x) is continuous for $a \le x < \infty$, then by definition

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx \qquad ....(i)$$

If there exists a finite limit on the right hand side of (i), then the improper integrals is said to be convergent; otherwise it is divergent.

Geometrically, the improper integral (i) for $f(x) > 0$, is the area of the figure bounded by the graph of the function $y = f(x)$, the straight line $x = a$ and the x-axis. Similarly,

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx \text{ and}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$

properties :

- $\int_0^a x f(x)\,dx = \frac{1}{2} a\int_0^a x f(x)$ if $f(a - x) = f(x)$ and $\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = \frac{a}{2}$
- $\int_0^{\pi/2} \log\sin x\,dx = \int_0^{\pi/2}\log\cos x\,dx$ $= -\frac{\pi}{2}\log 2 = \frac{\pi}{2}\log\frac{1}{2}$
- $\Gamma(n + 1) = n\,\Gamma(n), \Gamma(1) = 1, \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
- If m and n are non-negative integers, then

$$\int_0^{\pi/2}\sin^m x\cos^n x\,dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$

Reduction Formulae of some Define Integrals :

- $\int_0^{\infty} e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$
- $\int_0^{\infty} e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$
- $\int_0^{\infty} e^{-ax}x^n dx = \frac{n!}{a^{n+1}}$
- If $I_n = \int_0^{\pi/2}\sin^n x\,dx$, then

$$I_n = \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.....\frac{2}{3} & \text{(when n is odd)} \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.....\frac{1}{2}.\frac{\pi}{2} & \text{(when n is even)} \end{cases}$$

- If $I_n = \int_0^{\pi/2}\cos^n x\,dx$, then

$$I_m = \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.....\frac{2}{3} & \text{(when n is odd)} \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.....\frac{1}{2}.\frac{\pi}{2} & \text{(when n is even)} \end{cases}$$

Leibnitz's Rule :

If f(x) is continuous and u(x), v(x) are differentiable functions in the interval [a, b], then

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f\{v(x)\}\frac{d}{dx}v(x) - f\{u(x)\frac{d}{dx}u(x)$$

Summation of Series by Integration :

$$\lim_{n\to\infty}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right).\frac{1}{n} = \int_0^1 f(x)\,dx$$

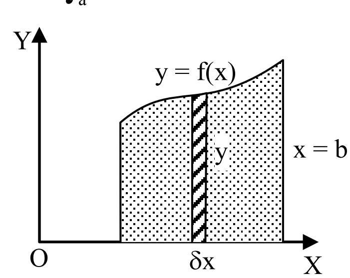
Some Important Results :

- $\sum_{r=0}^{n-1}\sin(\alpha + r\beta) = \frac{\sin\left\{\alpha+\frac{1}{2}(n-1)\beta\right\}\sin\left(\frac{1}{2}n\beta\right)}{\sin\left(\frac{1}{2}\beta\right)}$
- $\sum_{r=0}^{n-1}\cos(\alpha + r\beta) = \frac{\cos\left\{\alpha+\frac{1}{2}(n-1)\beta\right\}\sin\left(\frac{1}{2}n\beta\right)}{\sin\left(\frac{1}{2}\beta\right)}$
- $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - .... = \frac{\pi^2}{12}$
- $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... = \frac{\pi^2}{6}$

Area under Curves :

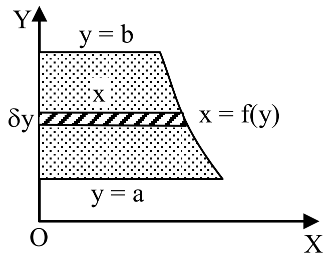
- Area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$

$$= \int_a^b y\,dx = \int_a^b f(x)\,dx$$



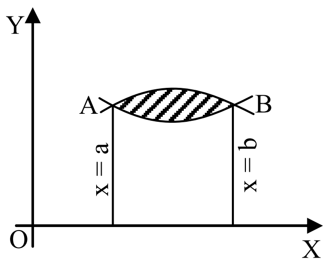
- Area bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = a$, $y = b$

$$= \int_a^b x\, dy = \int_a^b f(y)\, dy$$



- The area of the region bounded by $y_1 = f_1(x)$, $y_2 = f_2(x)$ and the ordinates x = a and x = b is given by

$$= \int_a^b f_2(x)\, dx - \int_a^b f_1(x)\, dx$$

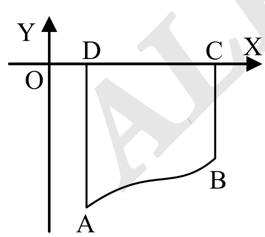


where $f_2(x)$ is y_2 of the upper curve and $f_1(x)$ is y_1 of the lower curve, i.e. the required area

$$= \int_a^b [f_2(x) - f_1(x)]\, dx = \int_a^b (y_2 - y_1)\, dx$$

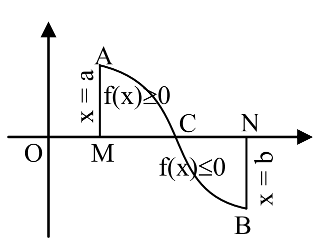
- $f(x) \le 0$ for all x in $a \le x \le b$, then area bounded by x-axis, the curve y = f(x) and the ordinates x = a, x = b is given by

$$= -\int_a^b f(x)\, dx$$



- If $f(x) \ge 0$ for $a \le x \le c$ and $f(x) \le 0$ for $c \le x \le b$, then area bounded by y = f(x), x-axis and the ordinates x = a, x = b is given by

$$= \int_a^c f(x)\, dx + \int_c^b -f(x)\, dx = \int_a^c f(x)\, dx - \int_c^b f(x)\, dx$$



16-DIFFERENTIAL EQUATIONS

Differential Equation :

An equation involving independent variable x, dependent variable y and the differential coefficients $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, ... is called differential equation.

Examples :

(1) $\frac{dy}{dx} = 1 + x + y$

(2) $\frac{dy}{dx} + xy = \cot x$

(3) $\left(\frac{d^4y}{dx^4}\right)^3 - 4\frac{dy}{dx} + 4y = 5 \cos 3x$

(4) $x^2\frac{d^2y}{dx^2} + \sqrt{1+\left(\frac{dy}{dx}\right)^2} = 0$

Order of a Differential Equation :

The order of a differential equation is the order of the highest derivative occurring in the differential equation. For example, the order of above differential equations are 1, 1, 4 and 2 respectively.

Degree of a Differential Equation :

The degree of the differential equation is the degree of the highest derivative when differential coefficients are free from radical and fraction. For example, the degree of above differential equations are 1, 1, 3 and 2 respectively.

Linear and Non-linear Differential Equation :

A differential equation in which the dependent variable and its differential coefficients occurs only in the first degree and are not multiplied together is called a linear differential equation. The general and n^{th} order differential equation is given below :

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_{n-1}\frac{dy}{dx} + a_n(x)y + \phi(x) = 0$$

Those equations which are not linear are called non-linear differential equations.

Formation of Differential Equation :

(1) Write down the given equation.

(2) Differentiate it successively with respect to x that number of times equal to the arbitrary constants.

(3) And hence on eliminating arbitrary constants results a differential equation which involves x, y, $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ .....

Solution of Differential Equation :

A solution of a differential equation is any function which when put into the equation changes it into an identity.

General and particular solution :

The solution which contains a number of arbitrary constant equal to the order of the equation is called general solution by giving particular values to the constants are called particular solutions.

Several Types of Differential Equations and their Solution :

(1) Solution of differential equation

$$\frac{dy}{dx} = f(x) \text{ is } y = \int f(x)dx + c$$

(2) Solution of differential equation

$$\frac{dy}{dx} = f(x)\, g(y) \text{ is } \int \frac{dy}{g(y)} = \int f(x)\,dx + c$$

(3) Solution of diff. equation $\frac{dy}{dx} = f(ax + by + c)$ by putting $ax + by + c = v$ and $\frac{dy}{dx} = \frac{1}{b}\left(\frac{dv}{dx} - a\right)$

$$\frac{dv}{a + bf(v)} = dx$$

Thus solution is by integrating

$$\int \frac{dv}{a + bf(v)} = \int dx$$

(4) To solve the homogeneous differential equation $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$, substitute $y = vx$ and so

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Thus $v + x\frac{dv}{dx} = f(v)$

$$\Rightarrow \frac{dx}{x} = \frac{dv}{f(v) - v}$$

Therefore solution is $\int \frac{dx}{x} = \int \frac{dv}{f(v) - v} + c$

Equation reducible to homogeneous form :

A differential equation of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2},$$

where $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, can be reduced to homogeneous form by adopting the following procedure :

Put $x = X + h,\ y = Y + k$,

so that $\frac{dY}{dX} = \frac{dy}{dx}$

The equation then transformed to

$$\frac{dY}{dX} = \frac{a_1X + b_1Y + (a_1h + b_1k + c_1)}{a_2X + b_2Y + (a_2h + b_2k + c_2)}$$

Now choose h and k such that $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$. Then for these values of h and k, the equation becomes

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

This is a homogeneous equation which can be solved by putting $Y = vX$ and then Y and X should be replaced by $y - k$ and $x - h$.

Special case :

If $\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$ and $\frac{a}{a'} = \frac{b}{b'} = m$ (say), i.e. when coefficient of x and y in numerator and denominator are proportional, then the above equation cannot be solved by the discussed before because the values of h and k given by the equations will be indeterminate.

In order to solve such equations, we proceed as explained in the following example.

Solve $\frac{dy}{dx} = \frac{2x - 6y + 7}{x - 3y + 4} = \frac{2(x - 3y) + 7}{x - 3y + 4}$

$\left\{\text{obviously } \frac{a}{a'} = \frac{b}{b'} = 2\right\}$

Put $x - 3y = v$

$\Rightarrow 1 - 3\frac{dy}{dx} = \frac{dv}{dx}$ (Now proceed yourself)

Solution of the linear differential equation :

$\frac{dy}{dx} + Py = Q$, where P and Q are either constants or functions of x, is

$$ye^{\int P\,dx} = \int \left(Qe^{\int P\,dx}\right) dx + c$$

Where $e^{\int P\,dx}$ is called the integrating factor.

Equations reducible to linear form :

- Bernoulli's equation : A differential equation of the form $\frac{dy}{dx} + Py = Qy^n$, where P and Q are functions of x alone is called Bernoulli's equation.

  Dividing by y^n, we get $y^{-n}\frac{dy}{dx} + y^{-(n-1)}.\ P = Q$

  Putting $y^{-(n-1)} = Y$, so that $\frac{(1-n)}{y^n}\frac{dy}{dx} = \frac{dY}{dx}$,

  we get $\frac{dY}{dx} + (1 - n)P.\ Y = (1 - n)Q$

  which is a linear differential equation.

- If the given equations is of the form

  $\frac{dy}{dx} + P.\ f(y) = Q\ .\ g(y)$, where P and Q are functions of x alone, we divide the equation by g(y) and get

  $$\frac{1}{g(y)}\frac{dy}{dx} + P.\frac{f(y)}{g(y)} = Q$$

  Now substitute $\frac{f(y)}{g(y)} = v$ and solve.

Solution of the differential equation :

$\frac{d^2y}{dx^2} = f(x)$ is obtained by integrating it with respect to x twice.

17-INVERSE TRIGNOMETRY

- Meaning of inverse function :
  1. $\sin\theta = x \Leftrightarrow \sin^{-1}x = \theta$
  2. $\cos\theta = x \Leftrightarrow \cos^{-1}x = \theta$
  3. $\tan\theta = x \Leftrightarrow \tan^{-1}x = \theta$
  4. $\cot\theta = x \Leftrightarrow \cot^{-1}x = \theta$
  5. $\sec\theta = x \Leftrightarrow \sec^{-1}x = \theta$
  6. $\text{cosec}\,\theta = x \Leftrightarrow \text{cosec}^{-1}x = \theta$
- Domains and Range of Functions :

Function	Domain	Range
$\sin^{-1}x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$\cos^{-1}x$	$-1 \le x \le 1$	$0 \le \theta \le \pi$
$\tan^{-1}x$	$-\infty < x < \infty$, i.e. $x \in R$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\text{cosec}^{-1}x$	$x \le -1, x \ge 1$	$\theta \ne 0, -\frac{\pi}{2} \le \theta < \frac{\pi}{2}$
$\sec^{-1}x$	$x \le -1, x \ge 1$	$\theta \ne \frac{\pi}{2}, 0 \le \theta \le \pi$
$\cot^{-1}x$	$-\infty < x < \infty$ i.e. $x \in R$	$0 < \theta < \pi$

- **Properties of Inverse Functions :**

(a) 1. $\sin^{-1}(\sin\theta) = \theta$, $\sin(\sin^{-1}x) = x$
2. $\cos^{-1}(\cos\theta) = \theta$, $\cos(\cos^{-1}x) = x$
3. $\tan^{-1}(\tan\theta) = \theta$, $\tan(\tan^{-1}x) = x$
4. $\cot^{-1}(\cot\theta) = \theta$, $\cot(\cot^{-1}x) = x$
5. $\sec^{-1}(\sec\theta) = \theta$, $\sec(\sec^{-1}x) = x$
6. $\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta$, $\text{cosec}(\text{cosec}^{-1}x) = x$

(b) 1. $\sin^{-1}x = \text{cosec}^{-1}(1/x)$
2. $\cos^{-1}x = \sec^{-1}(1/x)$
3. $\tan^{-1}x = \cot^{-1}(1/x)$

(c) 1. $\sin^{-1}(-x) = -\sin^{-1}x$
2. $\cos^{-1}(-x) = \pi - \cos^{-1}x$
3. $\tan^{-1}(-x) = -\tan^{-1}x$
4. $\cot^{-1}(-x) = \pi - \cot^{-1}x$
5. $\sec^{-1}(-x) = \pi - \sec^{-1}x$
6. $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$

(d). 1. $\sin^{-1}x + \cos^{-1}x = \pi/2$
2. $\tan^{-1}x + \cot^{-1}x = \pi/2$
3. $\sec^{-1}x + \text{cosec}^{-1}x = \pi/2$

- **Formulae for Sum and Difference of Inverse Function –**

1. $\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\frac{x+y}{1-xy} & \text{where } xy < 1 \\ \pi + \tan^{-1}\frac{x+y}{1-xy} & \text{when } xy > 1 \end{cases}$
2. $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$
3. $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right\}$
4. $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left\{xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right\}$
5. $\cot^{-1}x \pm \cot^{-1}y = \cot^{-1}\left[\frac{xy \mp 1}{y \pm x}\right]$
6. $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$

- **Some Important Results :**

1. $2\sin^{-1}x = \sin^{-1}2x\sqrt{1-x^2}$
2. $2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$
3. $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2} = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2}$
4. $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$
5. $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$
6. $3\tan^{-1}x = \tan^{-1}\frac{3x - x^3}{1-3x^2}$
7. $\tan^{-1}\left[\frac{x}{\sqrt{a^2-x^2}}\right] = \sin^{-1}\left(\frac{x}{a}\right)$
8. $\tan^{-1}\left[\frac{3a^2x - x^3}{a(a^2-3x^2)}\right] = 3\tan^{-1}\left(\frac{x}{a}\right)$
9. $\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right] = \frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2$

18-MATRICES AND DETERMINANTS

Matrices :

- An m × n matrix is a rectangular array of mn numbers (real or complex) arranged in an ordered set of m horizontal lines called rows and n vertical lines called columns enclosed in parentheses. An m × n matrix A is usually written as :

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & & & \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & & & & \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

Where $1 \le i \le m$ and $1 \le j \le n$
and is written in compact form as $A = [a_{ij}]_{m\times n}$

- A matrix $A = [a_{ij}]_{m \times n}$ is called
  (i) a rectangular matrix if $m \neq n$
  (ii) a square matrix if $m = n$
  (iii) a row matrix or row vector if $m = 1$
  (iv) a column matrix or column vector if $n = 1$
  (v) a null matrix if $a_{ij} = 0$ for all i, j and is denoted by $O_{m\times n}$
  (vi) a diagonal matrix if $a_{ij} = 0$ for $i \neq j$
  (vii) a scalar matrix if $a_{ij} = 0$ for $i \neq j$ and all diagonal elements a_{ii} are equal
- Two matrices can be added only when thye are of same order. If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, then sum of A and B is denoted by A + B and is a matrix $[a_{ij} + b_{ij}]_{m \times n}$
- The product of two matrices A and B, written as AB, is defined in this very order of matrices if number of columns of A (pre factor) is equal to the number of rows of B (post factor). If AB is defined , we say that A and B are conformable for multiplication in the order AB.

  If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$, then their product AB is a matrix $C = [c_{ij}]_{m \times p}$ where

  C_{ij} = sum of the products of elements of ith row of A with the corresponding elements of jth column of B.
- Types of matrices :
  (i) Idempotent if $A^2 = A$
  (ii) Periodic if $A^{k+1} = A$ for some positive integer k. The least value of k is called the period of A.
  (iii) Nilpotent if $A^k = O$ when k is a positive integer. Least value of k is called the index of the nilpotent matrix.
  (iv) Involutary if $A^2 = I$.
- The matrix obtained from a matrix $A = [a_{ij}]_{m \times n}$ by changing its rows into columns and columns of A into rows is called the transpose of A and is denoted by A'.
- A square matrix $a = [a_{ij}]_{n \times n}$ is said to be
  (i) Symmetric if $a_{ij} = a_{ji}$ for all i and j i.e. if $A' = A$.
  (ii) Skew-symmetric if
  $a_{ij} = -a_{ji}$ for all i and j i.e., if $A' = -A$.
- Every square matrix A can be uniquely written as sum of a symmetric and a skew-symmetric matrix.

  $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$ where $\frac{1}{2}(A + A')$ is symmetric and $\frac{1}{2}(A - A')$ is skew-symmetric.
- Let $A = [a_{ij}]_{m \times n}$ be a given matrix. Then the matrix obtained from A by replacing all the elements by their conjugate complex is called the conjugate of the matrix A and is denoted by $\overline{A} = [\overline{a}_{ij}]$.

Properties :

(i) $\overline{(\overline{A})} = A$

(ii) $\overline{(A+B)} = \overline{A} + \overline{B}$

(iii) $\overline{(\lambda A)} = \overline{\lambda}\,\overline{A}$, where λ is a scalar

(iv) $\overline{(AB)} = \overline{A}\,\overline{B}$.

Determinant :

Consider the set of linear equations $a_1x + b_1y = 0$ and $a_2x + b_2y = 0$, where on eliminating x and y we get the eliminant $a_1b_2 - a_2b_1 = 0$; or symbolically, we write in the determinant notation

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv a_1b_2 - a_2b_1 = 0$$

Here the scalar $a_1b_2 - a_2b_1$ is said to be the expansion of the 2 × 2 order determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ having 2 rows and 2 columns.

Similarly, a determinant of 3 × 3 order can be expanded as :

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$

$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$

$= \Sigma(\pm a_ib_jc_k)$

- To every square matrix $A = [a_{ij}]_{m \times n}$ is associated a number of function called the determinant of A and is denoted by $|A|$ or det A.

  Thus, $|A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$

- If $A = [a_{ij}]_{n \times n}$, then the matrix obtained from A after deleting ith row and jth column is called a submatrix of A. The determinant of this submatrix is called a minor or a_{ij}.
- Sum of products of elements of a row (or column) in a det with their corresponding cofactors is equal to the value of the determinant.

  i.e., $\sum_{i=1}^{n} a_{ij} C_{ij} = |A|$ and $\sum_{j=1}^{n} a_{ij} C_{ij} = |A|$.

- (i) If all the elements of any two rows or two columns of a determinant ate either identical or proportional, then the determinant is zero.

  (ii) If A is a square matrix of order n, then $|kA| = k^n |A|$.

  (iii) If Δ is determinant of order n and Δ' is the determinant obtained from Δ by replacing the elements by the corresponding cofactors, then

  $\Delta' = \Delta^{n-1}$

  (iv) Determinant of a skew-symmetric matrix of odd order is always zero.

- The determinant of a square matrix can be evaluated by expanding from any row or column.
- If $A = [a_{ij}]_{n \times n}$ is a square matrix and C_{ij} is the cofactor of a_{ij} in A, then the transpose of the matrix obtained from A after replacing each element by the corresponding cofactor is called the adjoint of A and is denoted by adj. A.

  Thus, adj. $A = [C_{ij}]'$.

  Properties of adjoint of a square matrix

  (i) If A is a square matrix of order n, then

  $A \cdot (\text{adj. } A) = (\text{adj. } A)A = |A| \ I_n$.

  (ii) If $|A| = 0$, then $A(\text{adj. } A) = (\text{adj. } A)A = O$.

  (iii) $|\text{adj. } A| = |A|^{n-1}$ if $|A| \neq 0$

  (iv) adj. (AB) = (adj. B) (adj. A).

  (v) adj. (adj. A) = $|A|^{n-2}$ A.

- Let A be a square matrix of order n. Then the inverse of A is given by $A^{-1} = \frac{1}{|A|}$ adj. A.
- Reversal law : If A, B, C are invertible matrices of same order, then

  (i) $(AB)^{-1} = B^{-1} A^{-1}$

  (ii) $(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$

- Criterion of consistency of a system of linear equations

  (i) The non-homogeneous system AX = B, B ≠ 0 has unique solution if $|A| \neq 0$ and the unique solution is given by $X = A^{-1}B$.

  (ii) Cramer's Rule : If $|A| \neq 0$ and $X = (x_1, x_2, \dots, x_n)'$ then for each i =1, 2, 3, …, n ; $x_i = \frac{|A_i|}{|A|}$ where Ai is the matrix obtained from A by replacing the ith column with B.

  (iii) If $|A| = 0$ and (adj. A) B = O, then the system AX = B is consistent and has infinitely many solutions.

  (iv) If $|A| = 0$ and (adj. A) B ≠ O, then the system AX = B is inconsistent.

  (v) If $|A| \neq 0$ then the homogeneous system AX = O has only null solution or trivial solution

  (i.e., $x_1 = 0, x_2 = 0, \dots x_n = 0$)

  (vi) If $|A| = 0$, then the system AX = O has non-null solution.

- (i) Area of a triangle having vertices at (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

  (ii) Three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear iff area of $\Delta ABC = 0$.

- A square matrix A is called an orthogonal matrix if AA′ = AA′ = I.
- A square matrix A is called unitary if $AA^\theta = A^\theta A = I$

  (i) The determinant of a unitary matrix is of modulus unity.

  (ii) If A is a unitary matrix then A', $\overline{A}$, A^θ, A^{-1} are unitary.

  (iii) Product of two unitary matrices is unitary.

- Differentiation of Determinants :

  Let $A = |C_1 \ C_2 \ C_3|$ is a determinant then

  $$\frac{dA}{dx} = |C'_1 \ C_2 \ C_3| + |C_1 \ C'_2 \ C_3| + |C_1 \ C_2 \ C'_3|$$

  Same process we have for row.

  Thus, to differentiate a determinant, we differentiate one column (or row) at a time, keeping others unchanged.

19-PROBABILITY

Some Definitions :

Experiment : A operation which can produce some well defined outcomes is known as an experiment.

Random experiment : If in each trail of an experiment conducted under identical conditions, the outcome is not unique, then such an experiment is called a random experiment.

Sample space : The set of all possible outcomes in an experiment is called a sample space. For example, in a throw of dice, the sample space is {1, 2, 3, 4, 5, 6}. Each element of a sample space is called a sample point.

Event :

An event is a subset of a sample space.

Simple event : An event containing only a single sample point is called an elementary or simple event. Events other than elementary are called composite or compound or mixed events.

For example, in a single toss of coin, the event of getting a head is a simple event.

Here S = {H, T} and E = {H}

In a simultaneous toss of two coins, the event of getting at least one head is a compound event.

Here S = {HH, HT, TH, TT} and E = {HH, HT, TH}

Equally likely events : The given events are said to be equally likely, if none of them is expected to occur in preference to the other.

Mutually exclusive events : If two or more events have no point in common, the events are said to be mutually exclusive. Thus E_1 and E_2 are mutually exclusive in $E_1 \cap E_2 = \phi$.

The events which are not mutually exclusive are known as compatible events.

Exhaustive events : A set of events is said to be totally exhaustive (simply exhaustive), if no event out side this set occurs and at least one of these event must happen as a result of an experiment.

Independent and dependent events : If there are events in which the occurrence of one does not depend upon the occurrence of the other, such events are known as independent events. On the other hand, if occurrence of one depend upon other, such events are known as dependent events.

Probability :

In a random experiment, let S be the sample space and $E \subseteq S$, then E is an event.

The probability of occurrence of event E is defined as

$$P(E) = \frac{\text{number of distinct elements in E}}{\text{number of distinct element in S}} = \frac{n(E)}{n(S)}$$

$$= \frac{\text{number of outocomes favourable to occurrence of E}}{\text{number of all possible outcomes}}$$

Notations :

Let A and B be two events, then

- $A \cup B$ or $A + B$ stands for the occurrence of at least one of A and B.
- $A \cap B$ or AB stands for the simultaneous occurrence of A and B.
- $A' \cap B'$ stands for the non-occurrence of both A and B.
- $A \subseteq B$ stands for "the occurrence of A implies occurrence of B".

Random variable :

A random variable is a real valued function whose domain is the sample space of a random experiment.

Bay's rule :

Let (H_j) be mutually exclusive events such that $P(H_j) > 0$ for j = 1, 2, ... n and $S = \bigcup_{j=1}^{n} H_j$. Let A be an events with $P(A) > 0$, then for j = 1, 2, ... , n

$$P\left(\frac{H_j}{A}\right) = \frac{P(H_j)P(A/H_j)}{\sum_{k=1}^{n} P(H_k)P(A/H_k)}$$

Binomial Distribution :

If the probability of happening of an event in a single trial of an experiment be p, then the probability of happening of that event r times in n trials will be ${}^nC_r\, p^r (1-p)^{n-r}$.

Some important results :

(A) • $P(A) = \frac{\text{Number of cases favourable to event A}}{\text{Total number of cases}}$

$$= \frac{n(A)}{n(S)}$$

- $P(\overline{A}) = \dfrac{\text{Number of cases not favourable to event A}}{\text{Total number of cases}}$

  $= \dfrac{n(\overline{A})}{n(S)}$

(B) Odd in favour and odds against an event : As a result of an experiment if "a" of the outcomes are favourable to an event E and b of the outcomes are against it, then we say that odds are a to b in favour of E or odds are b to a against E.

Thus odds in favour of an event E

$$= \frac{\text{Number of favourable cases}}{\text{Number of unfavourable cases}} = \frac{a}{b}$$

Similarly, odds against an event E

$$= \frac{\text{Number of unfavourable cases}}{\text{Number of favorable cases}} = \frac{b}{a}$$

Note :

- If odds in favour of an event are a : b, then the probability of the occurrence of that event is $\frac{a}{a+b}$ and the probability of non-occurrence of that event is $\frac{b}{a+b}$.
- If odds against an event are a : b, then the probability of the occurrence of that event is $\frac{b}{a+b}$ and the probability of non-occurrence of that event is $\frac{a}{a+b}$.

(C)
- $P(A) + P(\overline{A}) = 1$
- $0 \le P(A) \le 1$
- $P(\phi) = 0$
- $P(S) = 1$
- If $S = \{A_1, A_2, ... A_n\}$, then $P(A_1) + P(A_2) + ... + P(A_n) = 1$
- If the probability of happening of an event in one trial be p, then the probability of successive happening of that event in r trials is p^r.

(D)
- If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$ or $P(A + B) = P(A) + P(B)$
- If A and B are any two events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ or $P(A + B) = P(A) + P(B) - P(AB)$
- If A and B are two independent events, then $P(A \cap B) = P(A) . P(B)$ or $P(AB) = P(A) . P(B)$
- If the probabilities of happening of n independent events be $p_1, p_2, ... , p_n$ respectively, then

  (i) Probability of happening none of them $= (1 - p_1)(1 - p_2) .... (1 - p_n)$

  (ii) Probability of happening at least one of them $= 1 - (1 - p_1)(1 - p_2) .... (1 - p_n)$

  (iii) Probability of happening of first event and not happening of the remaining $= p_1(1 - p_2)(1 - p_3) .... (1 - p_n)$
- If A and B are any two events, then

  $P(A \cap B) = P(A) . P\left(\frac{B}{A}\right)$ or

  $P(AB) = P(A) . P\left(\frac{B}{A}\right)$

  Where $P\left(\frac{B}{A}\right)$ is known as conditional probability means probability of B when A has occured.
- **Difference between mutually exclusiveness and independence :** Mutually exclusiveness is used when the events are taken from the same experiment and independence is used when the events are taken from the same experiments.

(E)
- $P(A\overline{A}) = 0$
- $P(AB) + P(\overline{AB}) = 1$
- $P(\overline{A}B) = P(B) - P(AB)$
- $P(A\overline{B}) = P(A) - P(AB)$
- $P(A + B) = P(A\overline{B}) + P(\overline{A}B) + P(AB)$

Some important remark about coins, dice and playing cards :

- **Coins :** A coin has a head side and a tail side. If an experiment consists of more than a coin, then coins are considered to be distinct if not otherwise stated.
- **Dice :** A die (cubical) has six faces marked 1, 2, 3, 4, 5, 6. We may have tetrahedral (having four faces 1, 2, 3, 4,) or pentagonal (having five faces 1, 2, 3, 4, 5) die. As in the case of coins, If we have more than one die, then all dice are considered to be distinct if not otherwise stated.
- **Playing cards :** A pack of playing cards usually has 52 cards. There are 4 suits (Spade, Heart, Diamond and Club) each having 13 cards. There are two colours red (Heart and Diamond) and black (Spade and Club) each having 26 cards.

In thirteen cards of each suit, there are 3 face cards or coart card namely king, queen and jack. So there are in all 12 face cards (4 kings, 4 queens and 4 jacks). Also there are 16 honour cards, 4 of each suit namely ace, king, queen and jack.

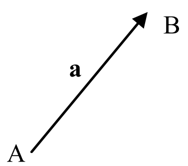
20-VECTORS

Representation of vectors :

Geometrically a vector is represent by a line segment. For example, $\mathbf{a} = \overrightarrow{AB}$. Here A is called the initial point and B, the terminal point or tip.

Magnitude or modulus of **a** is expressed as

$|\mathbf{a}| = |\overrightarrow{AB}| = AB.$



Types of Vector:

- **Zero or null vector :** A vector whose magnitude is zero is called zero or null vector and it is represented by $\vec{O}$.
- **Unit vector :** A vector whose modulus is unity, is called a unit vector. The unit vector in the direction of a vector **a** is denoted by $\hat{a}$, read as "a cap". Thus, $|\hat{a}| = 1$.

  $$\hat{a} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\text{Vector a}}{\text{Magnitude of a}}$$
- **Like and unlike vectors :** Vectors are said to be like when they have the same sense of direction and unlike when they have opposite directions.
- **Collinear or parallel vectors :** Vectors having the same or parallel supports are called collinear or parallel vectors.
- **Co-initial vectors :** Vectors having the same initial point are called co-initial vectors.
- **Coplanar vectors :** A system of vectors is said to be coplanar, if their supports are parallel to the same plane.

  Two vectors having the same initial point are always coplanar but such three or more vectors may or may not be coplanar.
- **Negative of a vector :** The vector which has the same magnitude as the vector **a** but opposite direction, is called the negative of **a** and is denoted by –**a**. Thus, if $\overrightarrow{PQ} = \mathbf{a}$, then $\overrightarrow{QP} = -\mathbf{a}$.

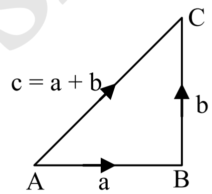
Properties of vectors :

(i) **Addition of vectors**

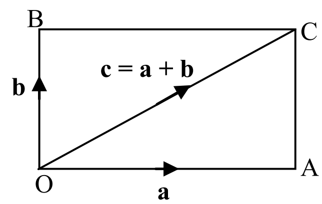
- **Triangle law of addition :** If in a ΔABC,

  $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$ and $\overrightarrow{AC} = \mathbf{c}$, **then**

  $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ i.e., $\mathbf{a} + \mathbf{b} = \mathbf{c}$



- **Parallelogram law of addition :** If in a parallelogram OACB, $\overrightarrow{OA} = a$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$



  Then $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$ i.e., $\mathbf{a} + \mathbf{b} = \mathbf{c}$, where OC is a diagonal of the parallelogram OABC.
- **Addition in component form :** If the vectors are defined in terms of **i**, **j**, and **k**, i.e., if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then their sum is defined as

  $\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}.$

Properties of vector addition :

Vector addition has the following properties.

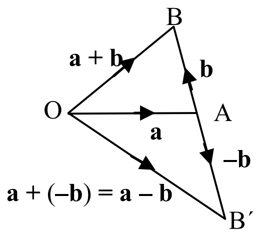
- **Binary operation :** The sum of two vectors is always a vector.
- **Commutativity :** For any two vectors **a** and **b**, $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.
- **Associativity :** For any three vectors **a, b** and **c**, $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
- **Identity :** Zero vector is the identity for addition. For any vector **a**, $\mathbf{0} + \mathbf{a} = \mathbf{a} = \mathbf{a} + \mathbf{0}$
- **Additive inverse :** For every vector **a** its negative vector –**a** exits such that $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$ i.e., (–**a**) is the additive inverse of the vector **a**.

Subtraction of vectors :

If **a** and **b** are two vectors, then their subtraction **a – b** is defined as **a – b** = **a** + (–**b**) where –**b** is the negative of **b** having magnitude equal to that of **b** and direction opposite to **b**. If

a = a_1**i** + a_2**j** + a_3**k**, **b** = b_1**i** + b_2**j** + b_3**k**

Then **a** – b = $(a_1 - b_1)$**i** + $(a_2 - b_2)$**j** + $(a_3 - b_3)$**k**.



Properties of vector subtraction :

(i) **a – b** ≠ **b – a**

(ii) (**a – b**) – **c** ≠ **a** – (**b – c**)

(iii) Since any one side of a triangle is less than the sum and greater than the difference of the other two sides, so for any two vectors a and b, we have

(a) |**a + b**| ≤ |**a**| + |**b**|

(b) |**a + b**| ≥ |**a**| – |**b**|

(c) |**a – b**| ≤ |**a**| + |**b**|

(d) |**a – b**| ≥ |**a**| – |**b**|

Multiplication of a vector by a scalar : If **a** is a vector and m is a scalar (i.e., a real number) then m**a** is a vector whose magnitude is m times that of **a** and whose direction is the same as that of **a**, if m is positive and opposite to that of **a**, if m is negative.

Properties of Multiplication of vector by a scalar : The following are properties of multiplication of vectors by scalars, for vector **a, b** and scalars m, n.

(i) m(–**a**) = (–m)**a** = –(m**a**)

(ii) (–m) (–**a**) = m**a**

(iii) m(n**a**) = (mn)**a** = n(m**a**)

(iv) (m + n)**a** = m**a** + n**a**

(v) m(**a** + **b**) = m**a** + m**b**

Position vector :

- $\overrightarrow{AB}$ **in terms of the position vectors of points A and B** : If **a** and **b** are position vectors of points A and B respectively. Then, $\overrightarrow{OA}$ = **a**, $\overrightarrow{OB}$ = **b**

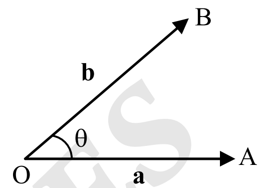
∴ $\overrightarrow{AB}$ = (Position vector of B)– (Position vector of A)

= $\overrightarrow{OB} - \overrightarrow{OA}$ = **b – a**

- **Position vector of a dividing point :** The position vectors of the points dividing the line AB in the ratio m : n internally or externally are $\frac{m\mathbf{b}+n\mathbf{a}}{m+n}$ or $\frac{m\mathbf{b}-n\mathbf{a}}{m-n}$.

Scalar or Dot product

Scalar or Dot product of two vectors : If **a** and **b** are two non-zero vectors and θ be the angle between them, then their scalar product (or dot product) is denoted by **a** . **b** and is defined as the scalar |**a**| |**b**| cosθ, where |**a**| and |**b**| are modulii of **a** and **b** respectively and 0 ≤ θ ≤ π. Dot product of two vectors is a scalar quantity.



Angle between two vectors : If **a, b** be two vectors inclined at an angle θ, then **a . b** = |**a**| |**b**| cos θ

$$\Rightarrow \cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)$$

If **a** = a_1**i** + a_2**j** + a_3**k** and **b** = b_1**i** + b_2**j** + b_3**k**; then

$$\theta = \cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}\right)$$

Properties of scalar product

- **Commutativity :** The scalar product of two vector is commutative i.e., **a . b = b . a**
- **Distributivity of scalar product over vector addition :** The scalar product of vectors is distributive over vector addition i.e.,

  (a) **a . (b + c) = a . b + a . c**, (Left distributivity)

  (b) (**b + c**) . **a** = **b . a + c . a**, (Right distributivity)
- Let **a** and **b** be two non-zero vectors **a . b** = 0

  ⇔ **a** ⊥ **b**.

  As **i, j, k** are mutually perpendicular unit vectors along the coordinate axes, therefore,

  i . j = j . i = 0; **j . k = k . j** = 0; **k . i = i . k** = 0.
- For any vector **a . a . a** = $|\mathbf{a}|^2$.

  As **i. j. k** are unit vectors along the co-ordinate axes, therefore **i . i** = $|\mathbf{i}|^2$ = 1, **j . j** = $|\mathbf{j}|^2$ = 1 and **k . k** = $|\mathbf{k}|^2$ = 1
- If m, n are scalars and **a . b** be two vectors, then m**a** . n**b** = mn(**a . b**) = (mn**a**) . **b** = **a** .(mn**b**)
- For any vectors **a** and **b**, we have

  (a) **a .** (–**b**) = –(**a . b**) = (–**a**). **b**

  (b) (–**a**) . (–**b**) = **a . b**
- For any two vectors **a** and **b**, we have

  (a) $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}.\mathbf{b}$

  (b) $|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a}.\mathbf{b}$

(c) $(\mathbf{a} + \mathbf{b}) \,.\, (\mathbf{a} - \mathrm{b}) = |\mathbf{a}|^2 - |\mathbf{b}|^2$

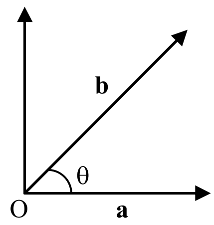
(d) $|\mathbf{a} + \mathbf{b}| = |\mathbf{a}| + |\mathbf{b}| \Rightarrow \mathbf{a} \,|\,|\, \mathbf{b}$

(e) $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 \Rightarrow \mathbf{a} \perp \mathbf{b}$

(f) $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}| \Rightarrow \mathbf{a} \perp \mathbf{b}$

Vector or Cross product

Vector product of two vectors : Let **a**, **b** be two non-zero, non-parallel vectors.



Then $\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin\theta\ \hat{\eta}$, and $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$, where θ is the angle between **a** and **b**, $\hat{\eta}$ is a unit vector perpendicular to the plane of **a** and **b** such that **a**, **b**, $\hat{\eta}$ form a right-handed system.

Properties of vector product :

- Vector product is not commutative i.e., if **a** and **b** are any two vectors, then $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$, however, $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$
- If **a**, **b** are two vectors and m, n are scalars, then $m\mathbf{a} \times n\mathbf{b} = mn(\mathbf{a} \times \mathbf{b}) = m(\mathbf{a} \times n\mathbf{b}) = n(m\mathbf{a} \times \mathbf{b})$.
- Distributivity of vector product over vector addition. Let **a**, **b**, **c** be any three vectors. Then

  (a) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (left distributivity)

  (b) $(\mathbf{b} + \mathbf{c}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a}$ (Right disributivity)
- For any three vectors **a**, **b**, **c** we have $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c}$.
- The vector product of two non-zero vectors is zero vector iff they are parallel (Collinear) i.e.,

  $\mathbf{a} \times \mathbf{b} = 0 \Leftrightarrow \mathbf{a} \,|\,|\, \mathbf{b}$, **a, b** are non-zero vectors.

  It follows from the above property that $\mathbf{a} \times \mathbf{a} = 0$ for every non-zero vector **a**, which in turn implies that $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$.
- Vector product of orthonormal triad of unit vectors **i, j, k** using the definition of the vector product, we obtain $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$, $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$, $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

Vector product in terms of components :

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$.

$$\text{Then, } \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Angle between two vectors :

If θ is the angle between **a** and **b** then $\sin\theta = \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|\,|\mathbf{b}|}$

Right handed system of vectors : Three mutually perpendicular vectors **a**, **b**, **c** form a right handed system of vector iff $\mathbf{a} \times \mathbf{b} = \mathbf{c}$, $\mathbf{b} \times \mathbf{c} = \mathbf{a}$, $\mathbf{c} \times \mathbf{a} = \mathbf{b}$

Left handed system of vectors : The vectors **a, b, c** mutually perpendicular to one another form a left handed system of vector iff $\mathbf{c} \times \mathbf{b} = \mathbf{a}$, $\mathbf{a} \times \mathbf{c} = \mathbf{b}$, $\mathbf{b} \times \mathbf{a} = \mathbf{c}$.

Area of parallelogram and triangle :

- The area of a parallelogram with adjacent sides **a** and **b** is $|\mathbf{a} \times \mathbf{b}|$.
- The area of a plane quadrilateral ABCD is $\frac{1}{2}|\overrightarrow{AC} \times \overrightarrow{BD}|$, where AC and BC are its diagonals.
- The area of a triangle ABC is $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$ or $\frac{1}{2}|\overrightarrow{BC} \times \overrightarrow{BA}|$ or $\frac{1}{2}|\overrightarrow{CB} \times \overrightarrow{CA}|$

Scalar triple product

Scalar triple product of three vectors : If **a**, **b**, **c** are three vectors, then scalar triple product is defined as the dot product of two vectors **a** and $\mathbf{b} \times \mathbf{c}$. It is generally denoted by $\mathbf{a} \,.\, (\mathbf{b} \times \mathbf{c})$ or $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

Properties of scalar triple product : If **a**, **b**, **c** are cyclically permuted, the value of scalar triple product remains the same. i.e.,

$$(\mathbf{a} \times \mathbf{b}) \,.\, \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \,.\, \mathbf{a} = (\mathbf{c} \times \mathbf{a}) .\, \mathbf{b}$$

or $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{b}\ \mathbf{a}\ \mathbf{c}] = [\mathbf{c}\ \mathbf{a}\ \mathbf{b}]$

Vector triple product

Let **a**, **b**, **c** be any three vectors, then the vectors $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ are called vector triple product of **a**, **b**, **c**.

Thus, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \,.\, \mathbf{c})\, \mathbf{b} - (\mathbf{a} \,.\, \mathbf{b})\mathbf{c}$

Properties of vector triple product :

- The vector triple product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is a linear combination of those two vectors which are within brackets.
- The vector $\mathbf{r} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is perpendicular to **a** and lies in the plane of **b** and **c**.
- The formula $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \,.\, \mathbf{c})\mathbf{b} - (\mathbf{a} \,.\, \mathbf{b})\mathbf{c}$ is true only when the vector outside the bracket is on the left most side. If it is not, we first shift on left by using the properties of cross product and then apply the same formula.

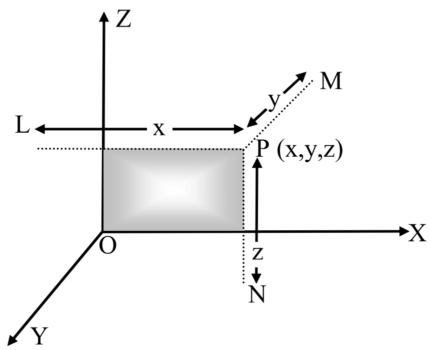
  Thus, $(\mathbf{b} \times \mathbf{c}) \times \mathbf{a} = -\{\mathbf{a} \times (\mathbf{b} \times \mathbf{c})\}$

  $= \{(\mathbf{a} \,.\, \mathbf{c})\mathbf{b} - (\mathbf{a} \,.\, \mathbf{b})\mathbf{c}\}$

  $= (\mathbf{a} \,.\, \mathbf{b})\mathbf{c} - (\mathbf{a} \,.\, \mathbf{c})\mathbf{b}$
- Vector triple product is a vector quantity.
- $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$

21-3-D GEOMETRY

Coordinates of a point :



x-coordinate = perpendicular distance of P from yz-plane

y-coordinate = perpendicular distance of P from zx-plane

z-coordinate = perpendicular distance of P from xy-plane

Coordinates of a point on the coordinate planes and axes:

yz-plane	:	$x = 0$
zx-plane	:	$y = 0$
xy-plane	:	$z = 0$
x-axis	:	$y = 0, z = 0$
y-axis	:	$y = 0, x = 0$
z-axis	:	$x = 0, y = 0$

Distance between two points :

If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points, then distance between them

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Coordinates of division point :

Coordinates of the point dividing the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $m_1 : m_2$ are

(i) in case of internal division

$$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right)$$

(ii) in case of external division

$$\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}, \frac{m_1z_2 - m_2z_1}{m_1 - m_2}\right)$$

Note: When m_1, m_2 are in opposite sign, then division will be external.

Coordinates of the midpoint:

When division point is the mid-point of PQ, then ration will be 1 : 1; hence coordinates of the mid-point of PQ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Coordinates of the general point :

The coordinates of any point lying on the line joining points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ may be taken as

$$\left(\frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}, \frac{kz_2 + z_1}{k+1}\right)$$

which divides PQ in the ratio k : 1. This is called general point on the line PQ.

Division by coordinate planes :

The ratios in which the line segment PQ joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is divided by coordinate planes are as follows :

(i) by yz-plane	:	$-x_1/x_2$ ratio
(ii) by zx-plane	:	$-y_1/y_2$ ratio
(iii) by xy-plane	:	$-z_1/z_2$ ratio

Coordinates of the centroid :

(i) If (x_1, y_1, z_1); (x_2, y_2, z_2) and (x_3, y_3, z_3) are vertices of a triangle then coordinates of its centroid are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

(ii) If (x_r, y_r, z_r); r = 1, 2, 3, 4 are vertices of a tetrahedron, then coordinates of its centroid are

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$$

Direction cosines of a line [Dc's] :

The cosines of the angles made by a line with positive direction of coordinate axes are called the direction cosines of that line.

Let α, β, γ be the angles made by a line AB with positive direction of coordinate axes then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of AB which are generally denoted by l, m, n. Hence

$$l = \cos\alpha,\ m = \cos\beta,\ n = \cos\gamma$$

x-axis makes 0º, 90º and 90º angles with three coordinate axes, so its direction cosines are cos 0º, cos 90º, cos 90º i.e. 1, 0, 0. Similarly direction cosines of y-axis and z-axis are 0, 1, 0 and 0, 0, 1 respectively. Hence

dc's of x-axis = 1, 0, 0

dc's of y-axis = 0, 1, 0

dc's of z-axis = 0, 0, 1

Relation between dc's

$\therefore\ l^2 + m^2 + n^2 = 1$

Direction ratios of a line [DR's] :

Three numbers which are proportional to the direction cosines of a line are called the direction ratios of that line. If a, b, c are such numbers which are proportional to the direction cosines l, m, n of a line then a, b, c are direction ratios of the line. Hence

$$\Rightarrow l = \pm \frac{a}{\sqrt{a^2+b^2+c^2}},$$

$$m = \pm \frac{b}{\sqrt{a^2+b^2+c^2}},\ n = \pm \frac{c}{\sqrt{a^2+b^2+c^2}}$$

Direction cosines of a line joining two points :

Let ≡ (x_1, y_1, z_1) and Q ≡ (x_2, y_2, z_2); then

(i) dr's of PQ : $(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)$

(ii)dc's of PQ : $\frac{x_2-x_1}{PQ}, \frac{y_2-y_1}{PQ}, \frac{z_2-z_1}{PQ}$

i.e., $\frac{x_2-x_1}{\sqrt{\Sigma(x_2-x_1)^2}}, \frac{y_2-y_1}{\sqrt{\Sigma(x_2-x_1)^2}}, \frac{z_2-z_1}{\sqrt{\Sigma(x_2-x_1)^2}}$

Angle between two lines :

Case I. When dc's of the lines are given

If l_1, m_1, and l_2,m_2 n_2 are dc's of given two lines, then the angle θ between them is given by

- $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$
- $\sin\theta = \sqrt{(\ell_1m_2 - \ell_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1\ell_2 - n_2\ell_1)^2}$

The value of sin θ can easily be obtained by the following form :

$$\sin\theta = \sqrt{\begin{vmatrix}\ell_1 & m_1\\ \ell_2 & m_2\end{vmatrix}^2 + \begin{vmatrix}m_1 & n_1\\ m_2 & n_2\end{vmatrix}^2 + \begin{vmatrix}n_1 & \ell_1\\ n_2 & \ell_2\end{vmatrix}^2}$$

Case II. When dr's of the lines are given

If a_1, b_1, c_1 and a_2, b_2, c_2 are dr's of given two lines, then the angle θ between them is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\sin\theta = \frac{\sqrt{\Sigma(a_1b_2 - a_2b_1)^2}}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Conditions of parallelism and perpendicularity of two lines :

Case I. When dc's of two lines AB and CD, say ℓ_1, m_1,n_1 and ℓ_2, m_2, n_2 are known

AB || CD ⇔ $\ell_1 = \ell_2$, $m_1 = m_2$, $n_1 = n_2$

AB ⊥ CD ⇔ $\ell_1\ \ell_2 + m_1m_2 + n_1n_2 = 0$.

Case II. When dr's of two lines AB and CD, say : a_1, b_1, c_1 and a_2, b_2, c_2 are known

AB || CD ⇔ $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

AB ⊥ CD ⇔ $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Area of a triangle :

Let A(x_1, y_1, z_1); B(x_2, y_2, z_2) and C(x_3, y_3, z_3) are vertices of a triangle. Then

dr's of AB = $x_2 - x_1, y_2 - y_1, z_2 - z_1$

= a_1, b_1, c_1 (say)

and AB = $\sqrt{a_1^2+b_1^2+c_1^2}$

dr's of BC = $x_3 - x_2, y_3 - y_2, z_3 - z_2$

= a_2, b_2, c_2(say)

and BC = $\sqrt{a_2^2+b_2^2+c_2^2}$

Now $\sin B = \frac{\sqrt{\Sigma(b_1c_2 - b_2c_1)^2}}{\sqrt{\Sigma a_1^2}\sqrt{\Sigma a_2^2}}$

$= \frac{\sqrt{\Sigma(b_1c_2 - b_2c_1)^2}}{AB.BC}$

∴ Area of ΔABC $= \frac{1}{2}$ AB. BC sin B

$= \frac{1}{2}\sqrt{\Sigma(b_1c_2 - b_2c_1)^2}$

Projection of a line segment joining two points on a line :

Let PQ be a line segment where P ≡ (x_1, y_1, z_1) and Q ≡ (x_2, y_2, z_2); and AB be a given line with dc's as l, m, n. If P'Q' be the projection of PQ on AB, then

P'Q' = PQ cos θ

where θ is the angle between PQ and AB. On replacing the value of cos θ in this, we shall get the following value of P'Q'.

P'Q' = $l\,(x_2 - x_1) + m(y_2 - y_1) + n\,(z_2 - z_1)$

Projection of PQ on x-axis : a = $|x_2 - x_1|$

Projection of PQ on y-axis : b = $|y_2 - y_1|$

Projection of PQ on z-axis : c = $|z_2 - z_1|$

Length of line segment PQ = $\sqrt{a^2+b^2+c^2}$

* If the given lines are $\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and $\frac{x-\alpha'}{\ell'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'}$, then condition for intersection is

- If the given lines are $\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ and $\frac{x-\alpha'}{\ell'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}$, then condition for intersections is

$$\begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ \ell & m & n \\ \ell & m' & n' \end{vmatrix}=0$$

Plane containing the above two lines is

$$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix}=0$$

Condition of coplanarity if both the lines are in general form:

Let the lines be

$ax + by + cz + d = 0 = a'x + b'y + c'z + d'$

and $\alpha x + \beta y + \gamma z + \delta = 0 = \alpha' x + \beta' y + \gamma' z + \delta'$

These are coplanar if $\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta' \end{vmatrix}=0$

Reduction of non-symmetrical form to symmetrical form:

Let equation of the line in non-symmetrical form be'

$a_1x + b_1y + c_1z + d_1 = 0;\ a_2x + b_2y + c_2z + d_2 = 0.$

To find the equation of the line in symmetrical form, we must know (i) its direction ratios (ii) coordinates of any point on it.

- **Direction ratios :** Let ℓ, m, n be the direction ratios of the line. Since the line lies in both the planes, it must be perpendicular to normals of both planes. So

$a_1\ell + b_1m + c_1n = 0;\ a_2\ell + b_2m + c_2n = 0$

From these equations, proportional values of ℓ, m, n can be found by cross-multiplication as

$$\frac{\ell}{b_1c_2-b_2c_1}=\frac{m}{c_1a_2-c_2a_1}=\frac{n}{a_1b_2-a_2b_1}$$

- **Point on the line :** Note that as ℓ, m, n cannot be zero simultaneously, so at least one must be non-zero. Let $a_1b_2 - a_2b_1 \neq 0$, then the line cannot be parallel to xy-plane, so it intersect it. Let it intersect xy-plane in $(x_1, y_1, 0)$. Then

$a_1x_1 + b_1y_1 + d_1 = 0$ and $a_2x_1 + b_2y_1 + d_2 = 0$

Solving these, we get a point on the line. Then its equation becomes

$$\frac{x-x_1}{b_1c_2-b_2c_1}=\frac{y-y_1}{c_1a_2-c_2a_1}=\frac{z-0}{a_1b_2-a_2b_1}$$

or
$$\frac{x-\frac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1}}{b_1c_2-b_2c_1}=\frac{y-\frac{d_1a_2-d_2a_1}{a_1b_2-a_2b_1}}{c_1a_2-c_2a_1}=\frac{z-0}{a_1b_2-a_2b_1}$$

Note : If $\ell \neq 0$, take a point on yz –plane as $(0, t_1, z_1)$ and if $m \neq 0$, take a point on xz-plane as $(x_1, 0, z_1)$

- **Skew lines :** The straight lines which are not parallel and non-coplanar i.e. non-intersecting are called skew lines.

If $\Delta = \begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} \neq 0$, the lines are skew.

Shortest distance : Suppose the equation of the lines are $\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$

and $\frac{x-\alpha'}{\ell'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}$. Then

$$\text{S.D.} = \frac{(\alpha-\alpha')(mn'-m'n)+(\beta-\beta')(n\ell'-n'\ell)(\ell m'-\ell' m)}{\sqrt{\Sigma(mn'-m'n)^2}}$$

$$= \begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix}$$

Some results for plane and straight line:

(i) General equation of a plane :

$ax + by + cz + d = 0$

where a, b, c are dr's of a normal to this plane.

(ii) Equation of a straight line :

General form : $\left.\begin{matrix} a_1x+b_1y+c_1z+d_1=0 \\ a_2x+b_2y+c_2z+d_2=0 \end{matrix}\right\}$

(In fact it is the straight line which is the intersection of two given planes)

Symmetric form : $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$

where (x_1, y_1, z_1) is a point on this line and a, b, c are its dr's

(iii) Angle between two planes :

If θ be the angle between planes $a_1x + b_1y\ \ c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, then

$$\cos\theta = \left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

(In fact angle between two planes is the angle between their normals.)

Further above two planes are

parallel $\Leftrightarrow \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

perpendicular $\Leftrightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$