7-TRIGNOMETRIC EQUATIONS

Functions with their Periods :

Function	**Period**
sin (ax + b), cos (ax + b), sec (ax + b), cosec (ax + b)	$2\pi/a$
tan(ax + b), cot (ax + b)	π/a
|sin (ax + b)|, |cos (ax + b)|, |sec (ax + b)|, |cosec (ax + b)|	π/a
|tan (ax + b)|, |cot (ax + b)|	$\pi/2a$

Trigonometrical Equations with their General Solution:

Trgonometrical equation	**General Solution**
$\sin\theta = 0$	$\theta = n\pi$
$\cos\theta = 0$	$\theta = n\pi + \pi/2$
$\tan\theta = 0$	$\theta = n\pi$
$\sin\theta = 1$	$\theta = 2n\pi + \pi/2$
$\cos\theta = 1$	$\theta = 2n\pi$
$\sin\theta = \sin\alpha$	$\theta = n\pi + (-1)^n \alpha$
$\cos\theta = \cos\alpha$	$\theta = 2n\pi \pm \alpha$
$\tan\theta = \tan\alpha$	$\theta = n\pi + \alpha$
$\sin^2\theta = \sin^2\alpha$	$\theta = n\pi \pm \alpha$
$\tan^2\theta = \tan^2\alpha$	$\theta = n\pi \pm \alpha$
$\cos^2\theta = \cos^2\alpha$	$\theta = n\pi \pm \alpha$
$\left.\begin{matrix}\sin\theta = \sin\alpha \\ \cos\theta = \cos\alpha\end{matrix}\right\|*$	$\theta = 2n\pi + \alpha$
$\left.\begin{matrix}\sin\theta = \sin\alpha \\ \tan\theta = \tan\alpha\end{matrix}\right\|*$	$\theta = 2n\pi + \alpha$
$\left.\begin{matrix}\tan\theta = \tan\alpha \\ \cos\theta = \cos\alpha\end{matrix}\right\|*$	$\theta = 2n\pi + \alpha$

***** If α be the least positive value of θ which satisfy two given trigonometrical equations, then the general value of θ will be $2n\pi + \alpha$.

Note :

1. If while solving an equation we have to square it, then the roots found after squaring must be checked whether they satisfy the original equation or not. e.g. Let x = 3. Squaring, we get $x^2 = 9$, $\therefore$ x = 3 and – 3 but x = – 3 does not satisfy the original equation x = 3.

2. Any value of x which makes both R.H.S. and L.H.S. equal will be a root but the value of x for which $\infty = \infty$ will not be a solution as it is an indeterminate form.

3. If xy = xz, then x(y – z) = 0 $\Rightarrow$ either x = 0 or y = z or both. But $\frac{y}{x} = \frac{z}{x} \Rightarrow y = z$ only and not x = 0, as it will make $\infty = \infty$. Similarly, if ay = az, then it will also imply y = z only as $a \neq 0$ being a constant.

Similarly, x + y = x + z $\Rightarrow$ y = z and x – y = x – z $\Rightarrow$ y = z. Here we do not take x = 0 as in the above because x is an additive factor and not multiplicative factor.

4. When $\cos\theta = 0$, then $\sin\theta = 1$ or –1. We have to verify which value of $\sin\theta$ is to be chosen which satisfies the equation $\cos\theta = 0 \Rightarrow \theta = \left(n + \frac{1}{2}\right)\pi$

If $\sin\theta = 1$, then obviously n = even. But if $\sin\theta = -1$, then n = odd.

Similarly, when $\sin\theta = 0$, then $\theta = n\pi$ and $\cos\theta = 1$ or –1.

If $\cos\theta = 1$, then n is even and if $\cos\theta = -1$, then n is odd.

5. The equations $a\cos\theta \pm b\sin\theta = c$ are solved as follows :

Put $a = r\cos\alpha$, $b = r\sin\alpha$ so that $r = \sqrt{a^2 + b^2}$ and $\alpha = \tan^{-1} b/a$.

The given equation becomes

$r[\cos\theta\cos\alpha \pm \sin\theta\sin\alpha] = c$;

$\cos(\theta \pm \alpha) = \frac{c}{r}$ provided $\left|\frac{c}{r}\right| \leq 1$.

Relation between the sides and the angle of a triangle:

1. **Sine formula :**

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{1}{2R}$$

Where R is the radius of circumcircle of triangle ABC.

2. **Cosine formulae :**

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \cos B = \frac{a^2+c^2-b^2}{2ac},$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

It should be remembered that, in a triangle ABC

- If $\angle A = 60^o$, then $b^2 + c^2 - a^2 = bc$
- If $\angle B = 60^o$, then $a^2 + c^2 - b^2 = ac$
- If $\angle C = 60^o$, then $a^2 + b^2 - c^2 = ab$

3. **Projection formulae :**

$a = b \cos C + c \cos B$, $b = c \cos A + a \cos C$

$c = a \cos B + b \cos A$

Trigonometrical Ratios of the Half Angles of a Triangle:

If $s = \frac{a+b+c}{2}$ in triangle ABC, where a, b and c are the lengths of sides of ΔABC, then

(a) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$, $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}$,

$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(b) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $\sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}$,

$\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(c) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$,

$\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$, $\tan\frac{C}{2} \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Napier's Analogy :

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}, \tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$$

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$$

Area of Triangle :

$$\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$$

$$\Delta = \frac{1}{2}\frac{a^2\sin B\sin C}{\sin(B+C)} = \frac{1}{2}\frac{b^2\sin C\sin A}{\sin(C+A)} = \frac{1}{2}\frac{c^2\sin A\sin B}{\sin(A+B)}$$

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc}$$

Similarly $\sin B = \frac{2\Delta}{ca}$ & $\sin C = \frac{2\Delta}{ab}$

Some Important Results :

1. $\tan\frac{A}{2}\tan\frac{B}{2} = \frac{s-c}{s}$ $\therefore \cot\frac{A}{2}\cot\frac{B}{2} = \frac{s}{s-c}$
2. $\tan\frac{A}{2} + \tan\frac{B}{2} = \frac{c}{s}\cot\frac{C}{2} = \frac{c}{\Delta}(s-c)$
3. $\tan\frac{A}{2} - \tan\frac{B}{2} = \frac{a-b}{\Delta}(s-c)$
4. $\cot\frac{A}{2} + \cot\frac{B}{2} = \frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\tan\frac{A}{2}\tan\frac{B}{2}} = \frac{c}{s-c}\cot\frac{C}{2}$
5. Also note the following identities :

- $\Sigma(p-q) = (p-q) + (q-r) + (r-p) = 0$
- $\Sigma p(q-r) = p(q-r) + q(r-p) + r(p-q) = 0$
- $\Sigma(p+a)(q-r) = \Sigma p(q-r) + a\Sigma(q-r) = 0$

Solution of Triangles :

1. **Introduction :** In a triangle, there are six elements viz. three sides and three angles. In plane geometry we have done that if three of the elements are given, at least one of which must be a side, then the other three elements can be uniquely determined. The procedure of determining unknown elements from the known elements is called solving a triangle.
2. **Solution of a right angled triangle :**

Case I. When two sides are given : Let the triangle be right angled at C. Then we can determine the remaining elements as given in the following table.

Given	**Required**
(i) a, b	$\tan A = \frac{a}{b}$, $B = 90^o - A$, $c = \frac{a}{\sin A}$
(ii) a, c	$\sin A = \frac{a}{c}$, $b = c\cos A$, $B = 90^o - A$

Case II. When a side and an acute angle are given – In this case, we can determine

Given	**Required**
(i) a, A	$B = 90^o - A$, $b = a\cot A$, $c = \frac{a}{\sin A}$
(ii) c, A	$B = 90^o - A$, $a = c\sin A$, $b = c\cos A$