

## 6-TRIGONOMETRIC RATIOS

### Some Important Definitions and Formulae :

**Measurement of angles :** The angles are measured in degrees, grades or in radius which are defined as follows:

**Degree :** A right angle is divided into 90 equal parts and each part is called a degree. Thus a right angle is equal to 90 degrees. One degree is denoted by  $1^\circ$ .

A degree is divided into sixty equal parts is called a minute. One minute is denoted by  $1'$ .

A minute is divided into sixty equal parts and each parts is called a second. One second is denoted by  $1''$ .

Thus,

1 right angle =  $90^\circ$  (Read as 90 degrees)

$1^\circ = 60'$  (Read as 60 minutes)

$1' = 60''$  (Read as 60 seconds).

**Grades :** A right angle is divided into 100 equal parts and each part is called a grade. Thus a right angle is equal to 100 grades. One grade is denoted by  $1^g$ .

A grade is divided into 100 equal parts and each part is called a minute and is denoted by  $1'$ .

A minute is divided into 100 equal parts and each part is called a second and is denoted by  $1''$

Thus,

1 right angled =  $100^g$  (Read as 100 grades)

$1^g = 100'$  (Read as 100 minutes)

$1' = 100''$  (Read as 100 seconds)

**Radians :** A radian is the angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.

### Domain and Range of a Trigonometric Function :

If  $f : X \rightarrow Y$  is a function, defined on the set  $X$ , then the domain of the function  $f$ , written as  $\text{Dom} f$  is the set of all independent variables  $x$ , for which the image  $f(x)$  is well defined element of  $Y$ , called the co-domain of  $f$ .

**Range** of  $f : X \rightarrow Y$  is the set of all images  $f(x)$  which belongs to  $Y$ , i.e.,

Range  $f = \{f(x) \in Y : x \in X\} \subseteq Y$

The domain and range of trigonometrical functions are tabulated as follows :

Trigo. Function	Domain	Range
$\sin x$	$\mathbb{R}$ , the set of all the real number	$-1 \leq \sin x \leq 1$
$\cos x$	$\mathbb{R}$	$-1 \leq \cos x \leq 1$
$\tan x$	$\mathbb{R} - \left\{ (2n+1)\frac{\pi}{2}, n \in \mathbb{I} \right\}$	$\mathbb{R}$
$\text{cosec } x$	$\mathbb{R} - \{n\pi, n \in \mathbb{I}\}$	$\mathbb{R} - \{x : -1 < x < 1\}$
$\sec x$	$\mathbb{R} - \left\{ (2n+1)\frac{\pi}{2}, n \in \mathbb{I} \right\}$	$\mathbb{R} - \{x : -1 < x < 1\}$
$\cot x$	$\mathbb{R} - \{n\pi, n \in \mathbb{I}\}$	$\mathbb{R}$

### Relation between Trigonometrically Ratios and identities:

- $\tan \theta = \frac{\sin \theta}{\cos \theta}$ ;  $\cot \theta = \frac{\cos \theta}{\sin \theta}$
- $\sin A \text{ cosec } A = \tan A \cot A = \cos A \sec A = 1$
- $\sin^2 \theta + \cos^2 \theta = 1$   
or  $\sin^2 \theta = 1 - \cos^2 \theta$  or  $\cos^2 \theta = 1 - \sin^2 \theta$
- $1 + \tan^2 \theta = \sec^2 \theta$   
or  $\sec^2 \theta - \tan^2 \theta = 1$  or  $\sec^2 \theta - 1 = \tan^2 \theta$
- $1 + \cot^2 \theta = \text{cosec}^2 \theta$   
or  $\text{cosec}^2 \theta - \cot^2 \theta = 1$  or  $\text{cosec}^2 \theta - 1 = \cot^2 \theta$
- Since  $\sin^2 A + \cos^2 A = 1$ , hence each of  $\sin A$  and  $\cos A$  is numerically less than or equal to unity. i.e.  
 $|\sin A| \leq 1$  and  $|\cos A| \leq 1$

or  $-1 \leq \sin A \leq 1$  and  $-1 \leq \cos A \leq 1$

**Note :** The modulus of real number  $x$  is defined as  $|x| = x$  if  $x \geq 0$  and  $|x| = -x$  if  $x < 0$ .

- Since  $\sec A$  and  $\text{cosec } A$  are respectively reciprocals of  $\cos A$  and  $\sin A$ , therefore the values of  $\sec A$  and  $\text{cosec } A$  are always numerically greater than or equal to unity i.e.

$\sec A \geq 1$  or  $\sec A \leq -1$

and  $\text{cosec } A \geq 1$  or  $\text{cosec } A \leq -1$

In other words, we never have

$-1 < \text{cosec } A < 1$  and  $-1 < \sec A < 1$ .

**Trigonometrical Ratios for Various Angles :**

$\theta$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$
$\sin \theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
$\tan \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\infty$	0	$\infty$	0

**Trigonometrical Ratios for Related Angles :**

$\theta$	$-\theta$	$\frac{\pi}{2} \pm \theta$	$\pi \pm \theta$	$\frac{3\pi}{2} \pm \theta$	$2\pi \pm \theta$
$\sin$	$-\sin \theta$	$\cos \theta$	$\mp \sin \theta$	$-\cos \theta$	$\pm \sin \theta$
$\cos$	$\cos \theta$	$\mp \sin \theta$	$-\cos \theta$	$\pm \sin \theta$	$\cos \theta$
$\tan$	$-\tan \theta$	$\mp \cot \theta$	$\pm \tan \theta$	$\mp \cot \theta$	$\pm \tan \theta$
$\cot$	$-\cot \theta$	$\mp \tan \theta$	$\pm \cot \theta$	$\mp \tan \theta$	$\pm \cot \theta$

**Addition and Subtraction Formulae :**

- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
- $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
- $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
- $\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$
- $\sin(A + B) \sin(A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$
- $\cos(A + B) \cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

**Formulae for Changing the Sum or Difference into Product :**

- $\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$
- $\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}$
- $\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$
- $\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}$

**Formulae for Changing the Product into Sum or Difference :**

- $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$
- $2 \cos A \sin B = \sin(A + B) - \sin(A - B)$
- $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$
- $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

**Formulae Involving Double, Triple and Half Angles :**

- $\sin 2\theta = 2 \sin \theta \cos \theta = \frac{2 \tan \theta}{1 + \tan^2 \theta}$
- $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$
- $\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}$ ;  $\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}$
- $\tan \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}$
- $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$
- $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$   
or  $\sin^3 \theta = \frac{1}{4} (3 \sin \theta - \sin 3\theta)$
- $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$   
or  $\cos^3 \theta = \frac{1}{4} (3 \cos \theta + \cos 3\theta)$
- $\tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} \left[ \theta \neq n\pi + \frac{\pi}{6} \right]$

**Trigonometrical Ratios for Some Special Angles :**

$\theta$	$7\frac{1}{2}^\circ$	$15^\circ$	$22\frac{1}{2}^\circ$
$\sin \theta$	$\frac{\sqrt{4 - \sqrt{2} - \sqrt{6}}}{2\sqrt{2}}$	$\frac{\sqrt{3} - 1}{2\sqrt{2}}$	$\frac{\sqrt{2} - \sqrt{2}}{2}$
$\cos \theta$	$\frac{\sqrt{4 + \sqrt{2} + \sqrt{6}}}{2\sqrt{2}}$	$\frac{\sqrt{3} + 1}{2\sqrt{2}}$	$\frac{\sqrt{2} + \sqrt{2}}{2}$
$\tan \theta$	$\frac{(\sqrt{3} - \sqrt{2})}{(\sqrt{2} - 1)}$	$2 - \sqrt{3}$	$\sqrt{2} - 1$

$\theta$	$18^\circ$	$36^\circ$
$\sin \theta$	$\frac{\sqrt{5} - 1}{4}$	$\frac{\sqrt{10 - 2\sqrt{5}}}{4}$
$\cos \theta$	$\frac{\sqrt{10 + 2\sqrt{5}}}{4}$	$\frac{\sqrt{5} + 1}{4}$
$\tan \theta$	$\frac{\sqrt{25 - 10\sqrt{5}}}{5}$	$\sqrt{5 - 2\sqrt{5}}$

**Important Points to Remember :**

- Maximum and minimum values of  $a \sin x + b \cos x$  are  $+\sqrt{a^2 + b^2}$ ,  $-\sqrt{a^2 + b^2}$  respectively.

- $\sin^2 x + \operatorname{cosec}^2 x \geq 2$  for every real  $x$ .
- $\cos^2 x + \sec^2 x \geq 2$  for every real  $x$ .
- $\tan^2 x + \cot^2 x \geq 2$  for every real  $x$
- If  $x = \sec \theta + \tan \theta$ , then  $\frac{1}{x} = \sec \theta - \tan \theta$
- If  $x = \operatorname{cosec} \theta + \cot \theta$ , then  $\frac{1}{x} = \operatorname{cosec} \theta - \cot \theta$
- $\cos \theta \cdot \cos 2\theta \cdot \cos 4\theta \cdot \cos 8\theta$

$$\dots \cos 2^{n-1}\theta = \frac{\sin 2^n \theta}{2^n \sin \theta}$$

- $\sin \theta \sin (60^\circ - \theta) \sin (60^\circ + \theta) = \frac{1}{4} \sin 3\theta$
- $\cos \theta \cos (60^\circ - \theta) \cos (60^\circ + \theta) = \frac{1}{4} \cos 3\theta$
- $\tan \theta \tan (60^\circ - \theta) \tan (60^\circ + \theta) = \tan 3\theta$

#### Conditional Identities :

##### 1. If $A + B + C = 180^\circ$ , then

- $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$
- $\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C$
- $\sin (B + C - A) + \sin (C + A - B) + \sin (A + B - C) = 4 \sin A \sin B \sin C$
- $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$
- $\cos 2A + \cos 2B - \cos 2C = 1 - 4 \sin A \sin B \cos C$

##### 2. If $A + B + C = 180^\circ$ , then

- $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$
- $\sin A + \sin B - \sin C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$
- $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$
- $\cos A + \cos B - \cos C = -1 + 4 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$
- $\frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B} = 2$

##### 3. If $A + B + C = \pi$ , then

- $\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \cos C$
- $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$
- $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C$
- $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2 \sin A \sin B \cos C$

##### 4. If $A + B + C = \pi$ , then

- $\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} = 1 - 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$
- $\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} = 2 + 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

- $\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} - \sin^2 \frac{C}{2} = 1 - 2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$
- $\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} - \cos^2 \frac{C}{2} = 2 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$

##### 5. If $x + y + z = \pi/2$ , then

- $\sin^2 x + \sin^2 y + \sin^2 z = 1 - 2 \sin x \sin y \sin z$
- $\cos^2 x + \cos^2 y + \cos^2 z = 2 + 2 \sin x \sin y \sin z$
- $\sin 2x + \sin 2y + \sin 2z = 4 \cos x \cos y \cos z$

##### 6. If $A + B + C = \pi$ , then

- $\tan A + \tan B + \tan C = \tan A \tan B \tan C$
- $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$
- $\tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} + \tan \frac{A}{2} \tan \frac{B}{2} = 1$
- $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$

##### 7. (a) For any angles $A, B, C$ we have

- $\sin (A + B + C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$
- $\cos (A + B + C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$
- $\tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

##### (b) If $A, B, C$ are the angles of a triangle, then

$$\sin(A + B + C) = \sin \pi = 0 \text{ and}$$

$$\cos(A + B + C) = \cos \pi = -1$$

then (a) gives

$$\sin A \sin B \sin C$$

$$= \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C$$

and (a) gives

$$1 + \cos A \cos B \cos C$$

$$= \cos A \sin B \sin C + \sin A \cos B \sin C + \sin A \sin B \cos C$$

#### Method of Componendo and Dividendo :

If  $\frac{p}{q} = \frac{a}{b}$ , then by componendo and dividendo we can write

$$\frac{p-q}{p+q} = \frac{a-b}{a+b} \text{ or } \frac{q-p}{q+p} = \frac{b-a}{b+a}$$

$$\text{or } \frac{p+q}{p-q} = \frac{a+b}{a-b} \text{ or } \frac{q+p}{q-p} = \frac{b+a}{b-a}$$