4-BINOMIAL THEOREM

Binomial Theorem (For a positive Integral Index) :

If n is a positive integer and x, a are two real or complex quantities, then

$(x + a)^n = {}^nC_0\, x^n + {}^nC_1 x^{n-1}\, a + {}^nC_2\, x^{n-2}\, a^2 + \ldots + {}^nC_r x^{n-r}\, a^r + \ldots + {}^nC_{n-1} x\, a^{n-1} + {}^nC_n a^n$ ..(1)

The coefficient ${}^nC_0, {}^nC_1, \ldots, {}^nC_n$ are called binomial coefficients.

Properties of Binomial Expansion :

- There are (n + 1) terms in the expansion of $(x + a)^n$, n being a positive integer.
- In any term of expansion (1), the sum of the exponents of x and a is always constant = n.
- The binomial coefficients of term equidistant from the beginning and the end are equal, i.e. ${}^nC_r = {}^nC_{n-r}$ $(0 \le r \le n)$.
- The general term of the expansion is $(r + 1)^{th}$ term usually denoted by $T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$ $(0 \le r \le n)$.
- The middle term in the expansion of $(x + a)^n$

(a) If n is even then there is just one middle term, i.e. $\left(\frac{n}{2}+1\right)^{th}$ term.

(b) if n is odd, then there are two middle terms, i.e. $\left(\frac{n}{2}+1\right)^{th}$ term and $\left(\frac{n+3}{2}\right)^{th}$ term.

- The greatest term in the expansion of $(x + a)^n$, $x, a \in R$ and $x, a > 0$ can be obtained as below :

$$\therefore \quad \frac{T_{r+1}}{T_r} = \frac{n-r+1}{r}\frac{a}{x}$$

$$\text{or} \quad \frac{T_{r+1}}{T_r} - 1 = \frac{(n+1)a - r(a+x)}{rx}$$

$$= \frac{(a+x)}{rx}\left\{\frac{(n+1)a}{a+x} - r\right\} = \frac{a+x}{rx}|k - r|,$$

where $k = \frac{(n+1)a}{a+x}$

Now, suppose that

(i) $k = \frac{(n+1)a}{a+x}$ is an integer. We have

$$T_{r+1} > T_r \Leftrightarrow \frac{T_{r+1}}{T_r} > 1 \Leftrightarrow r < k \text{ (i.e. } 1 \le r < k)$$

Along , $T_{r+1} = T_r \Leftrightarrow \frac{T_{r+1}}{T_r} = 1 \Leftrightarrow r = k,$

i.e. $T_{k+1} = T_k > T_{k-1} > \ldots > T_3 > T_2 > T_1$

In this case there are two greatest terms T_k and T_{k+1}.

(ii) $k = \frac{(n+1)a}{a+x}$ is not an integer. Let [k] be the greatest integer in k. We have

$T_{r+1} > T_r \Leftrightarrow \frac{T_{r+1}}{T_r} \Leftrightarrow r < k = [k] + \text{(fraction)}$

$\Leftrightarrow r \le [k]$

i.e. $T_1 < T_2 < T_3 < \ldots < T_{[k]-1} < T_{[k]} < T_{[k]+1}$

In this case there is exactly one greatest term viz. $([k] + 1)^{th}$ term.

- **Term independent of** x in the expansion of $(x + a)^n$ – Let T_{r+1} be the term independent of x. Equate to zero the index of x and you will find the value of r.
- The number of term in the expansion of $(x + y + z)^n$ is $\frac{(n+1)(n+2)}{2}$, where n is a positive integer.

Pascal Triangle

In$(x + a)^n$ when expanded the various coefficients which occur are ${}^nC_0, {}^nC_1, {}^nC_2, \ldots$ The Pascal triangle gives the values of these coefficients for n = 0, 1, 2, 3, 4, 5, ...

n = 0	1
n = 1	1 1
n = 2	1 2 1
n = 3	1 3 3 1
n = 4	1 4 6 4 1
n = 5	1 5 10 10 5 1
n = 6	1 6 15 20 15 6 1
n = 7	1 7 21 35 35 21 7 1
n = 8	1 8 28 56 70 56 28 8 1

Rule : It is to be noted that the first and least terms in each row is 1. The terms equidistant from the beginning and end are equal. Any number in any row is obtained by adding the two numbers in the preceding row which are just at the left and just at the right of the given number, e.g. the number 21 in the row for n = 7 is the sum of 6 (left) and 15 (right) which occur in the preceding row for n = 6.

Important Cases of Binomial Expansion :

- If we put x = 1 in (1), we get
  $(1 + a)^n = {}^nC_0 + {}^nC_1a + {}^nC_2a^2 + ...$
  $+ {}^nC_ra^r + ... {}^nC_na^n$ ...(2)
- If we put x = 1 and replace a by – a, we get
  $(1 - a)^n = {}^nC_0 - {}^nC_1a + {}^nC_2a^2 - ....$
  $+ (-1)^r\, {}^nC_ra^r + ... + (-1)^n\, {}^nC_na^n$ ...(3)
- Adding and subtracting (2) and (3), and then dividing by 2, we get
  $\frac{1}{2}\{(1 + a)^n + (1 - a)^n\} = {}^nC_0 + {}^nC_2a^2 + {}^nC_4a^4 + ...$ ...(4)
  $\frac{1}{2}\{(1 + a)^n - (1 - a)^n\} = {}^nC_1a + {}^nC_3a^3 + {}^nC_5a^5 + ...$ ...(5)

Properties of Binomial Coefficients :

If we put a = 1 in (2) and (3), we get

$2^n = {}^nC_0 + {}^nC_2 + ... + {}^nC_r + ...+ {}^nC_r + ... {}^nC_{n-1} + {}^nC_n$

and $0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - ...... + ... + (-1)^n\, {}^nC_n$

$\therefore\ {}^nC_0 + {}^nC_2 + ... = {}^nC_1 + {}^nC_3 + ... = \frac{1}{2}[2^n \pm 0]$

$= 2^{n-1}$ ...(6)

Due to convenience usually written as

$C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$

and $C_0 + C_1 + C_2 + C_3 + ... + C_n = 2^n$

Where ${}^nC_r \equiv C_r = \frac{n!}{r!(n-r)!}$

$= \frac{n(n-1)(n-2)...(n-r+1)}{r!}$

Some other properties to remember :

- $C_1 + 2C_2 + 3C_3 + ... + nC_n = n \,.\, 2^{n-1}$
- $C_1 - 2C_2 + 3C_3 - .... = 0$
- $C_0 + 2C_1 + 3C_2 + ... + (n + 1)C_n = (n + 2)\, 2^{n-1}$
- $C_0C_r + C_1C_{r+1} + ... + C_{n-r}C_n = \frac{(2n)!}{(n-r)!.(n+r)!}$
- $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = \frac{(2n)!}{(n!)^2}$
- $C_0^2 - C_1^2 + C_2^2 - C_3^2 + ...$
  $= \begin{cases} 0, & \text{if n is odd} \\ (-1)^{n/2}.\,{}^nC_{n/2}, & \text{if n is even} \end{cases}$

Binomial Theorem for Any Index :

- The binomial theorem for any index states that
  $(1 + x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$ ...(7)
  Where n is any index (positive or negative)
- The general term in expansion (7) is
  $T_{r+1} = \frac{n(n-1)......(n-r+1)}{r!}x^r$
- In this expansion there are infinitely many terms.
- This expansion is valid for |x| < 1 and first term unity.
- When x is small compared with 1, we see that the terms finally get smaller and smaller. If x is very small compared with 1, we take 1 as a first approximation to the value of $(1 + x)^n$ or 1 + nx as a second approximation.
- Replacing n by – n in the above expansion, we get

$(1 + x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + ... + (-1)^r\frac{n(n+1)(n+2)...(n+r-1)}{r!}x^r + ...$

Replacing x by – x in this expansion, we get

$(1 - x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + ... + \frac{n(n+1)(n+2)...(n+r-1)}{r!}x^r + ...$

Important expansions for n = –1, –2 are :

- $(1 + x)^{-1} = 1 - x + x^2 - x^3 + ...+ (-1)^rx^r + ...$ to ∞
- $(1 - x)^{-1} = 1 + x + x^2 + x^3 + ... + x^r + ...$ to ∞
- $(1 + x)^{-2} = 1 - 2x + 3x^2 - .... + (-1)^r(r + 1)x^r + ...$ to ∞
- $(1 - x)^{-2} = 1 + 2x + 3x^2 + ... + (r + 1)x^r + ...$ to ∞
- $(1 + x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + ... + (-1)^r\frac{(r+1)(r+2)}{2!}x^r + ...$
- $(1 - x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + ... + \frac{(r+1)(r+2)}{2!}x^r + ...$