3-PROGRESSION AND MATHEMATICAL INDUCTION

Arithmetic Progression (AP)

AP is a progression in which the difference between any two consecutive terms is constant. This constant difference is called **common difference** (c.d.) and generally it is denoted by d.

Standard form: Its standard form is

$a + (a + d) + (a + 2d) + ...$

General term :

$T_n = a + (n - 1) d$

If $T_n = l$ then it should be noted that

(i) $d = \frac{\ell - a}{n-1}$ (ii) $n = 1 + \frac{\ell - a}{d}$

Note: a, b, c are in AP $\Leftrightarrow 2b = a + c$

Sum of n terms of an AP :

$$S_n = \frac{n}{2}(a + \ell)$$

where l is last term (nth term). Replacing the value of l, it takes the form

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Arithmetic Mean :

(i) If A be the AM between two numbers a and b, then $A = \frac{1}{2}(a + b)$

(ii) The AM of n numbers $a_1, a_2, ..., a_n$

$= \frac{1}{n}(a_1 + a_2 + ... + a_n)$

(iii) n AM's between two numbers

If $A_1, A_2, ..., A_n$ be n AM's between a and b then

$a\ A_1, A_2, ..., A_n, b$ is an AP of $(n + 2)$ terms. Its common difference d is given by

$$T_{n+2} = b = a + (n + 1)d \Rightarrow d = \frac{b-a}{n+1}$$

so $A_1 = a + d$, $A_2 = a + 2d, ..., A_n = a + nd$.

Sum of n AM's between a and b

$\therefore\ \Sigma A_n = n(A)$

Assuming numbers in AP :

(i) When number of terms be odd

Three terms : $a - d, a, a + d$

Five terms : $a - 2d, a - d, a, a + d, a + 2d$

....

(ii) When number of terms be even

Four terms: $a - 3d, a - d, a + d, a + 3d$

Six terms : $a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$

....

Geometrical Progression (GP) :

A progression is called a GP if the ratio of its each term to its previous term is always constant. This constant ratio is called its **common ratio** and it is generally denoted by r.

Standard form : Its standard form is

$a + ar + ar^2 + ...$

General term : $T_n = ar^{n-1}$

a, b, c are in GP $\Leftrightarrow \frac{b}{a} = \frac{c}{b} \Leftrightarrow b^2 = ac$

Sum of n terms of a GP :

The sum of n terms of a GP $a + ar + ar^2 + ...$ is given by

$$S_n = \begin{cases} \frac{a(1-r^n)}{1-r} = \frac{a - \ell r}{1-r}, \text{when } r < 1 \\ \frac{a(r^n - 1)}{r-1} = \frac{\ell r - a}{r-1}, \text{when } r > 1 \end{cases}$$

when $\ell = T_n$.

Sum of an infinite GP :

(i) When $r > 1$, then $r^n \to \infty$, so $S_n \to \infty$ Thus when $r > 1$, the sum S of infinite GP $= \infty$

(ii) When $|r| < 1$, then $r^n \to 0$, so

$$S = \frac{a}{1-r}$$

(iii) When $r = 1$, then each term is a so $S = \infty$.

Geometric Mean :

(i) If G be the GM between a and b then

$G = \sqrt{ab}$

(ii) G.M. of n numbers $a_1, a_2 ......, a_n = (a_1a_2a_3 .....a_n)^{1/n}$

(iii) n GM's between two numbers

$\Rightarrow r = (b/a)^{1/n+1}$

Product of n GM's between a and b

Product of GM's = $(ab)^{n/2} = G^n$

Assuming numbers in GP :

(i) When number of terms be odd

Three terms : a/r, a, ar

Five terms : a/r^2, a/r, a, ar, ar^2

....

(ii) When number of terms be even

Four terms : a/r^3, a/r, ar, ar^3

Six terms : a/r^5, a/r^3, a/r, ar, ar^3, ar^5

Arithmetic-Geometric Progression :

If each term of a progression is the product of the corresponding terms of an AP and a GP, then it is called arithmetic-geometric progression (AGP). For example:

a, (a + d)r, $(a + 2d)r^2$ ....

$T_n = [a + (n-1)d]\, r^{n-1}$

$$S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r}$$

$$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2} \qquad |r| < 1$$

Harmonic Progression :

A progression is called a harmonic progression (HP) if the reciprocals of its terms are in AP.

Standard form : $\frac{1}{a} + \frac{1}{a+d} + \frac{1}{a+2d} + ...$

General term : $T_n = \frac{1}{a+(n-1)d}$

$\therefore$ a, b, c are in HP $\Leftrightarrow \frac{2}{b} = \frac{1}{a} + \frac{1}{c} \Leftrightarrow b = \frac{2ac}{a+c}$

Harmonic Mean :

(i) If H be a HM between two numbers a and b, then

$$H = \frac{2ab}{a+b} \text{ or } \frac{2}{H} = \frac{1}{a} + \frac{1}{b}$$

(ii) To find n HM's between a and b we first find n AM's between 1/a and 1/b, then their reciprocals will be the required HM's.

Relations between AM, GM and HM :

$G^2 = AH$

A > G > H, when a, b > 0.

If A and AM and GM respectively between two positive numbers, then those numbers are

$$A + \sqrt{A^2 - G^2}, A - \sqrt{A^2 - G^2}$$

Some Important Results :

- If number of terms in an AP/GP/HP is odd then its mid term is the AM/GM/HM between the first and last term.
- If number of terms in an AP/GP/HP is even the AM/GM/HM of its two middle terms is equal to the AM/GM/HM between the first and last term.
- a, b, c are in AP, GP and HP $\Leftrightarrow$ a = b = c
- a, b, c are in AP and HP $\Rightarrow$ a, b,c are in GP.
- a, b, c are in AP

  $\Leftrightarrow \frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in AP. $\Leftrightarrow$ bc, ca, ab are in HP.
- a, b, c are in GP $\Leftrightarrow a^2, b^2, c^2$ are in GP.
- a, b, c are in GP $\Leftrightarrow$ loga, logb, logc are in AP.
- a, b, c are in GP $\Leftrightarrow \log_a m\ \log_b m, \log_c m$ are in HP.
- a, b, c d are in GP $\Leftrightarrow$ a + b, b + c, c + d are in GP.
- a, b, c are in AP $\Leftrightarrow \alpha^a, \alpha^b, \alpha^c$ are in GP ($\alpha \in R_0$)

Principle of Mathematical Induction :

It states that any statement P(n) is true for all positive integral values of n if

(i) P(1) is true i.e., it is true for n = 1.

(ii) P(m) is true $\Rightarrow$ P(m + 1) is also true

i.e., if the statement is true for n = m then it must also be true for n = m + 1.

Some Formula based on the Principle of Induction :

- $\Sigma n = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

  (Sum of first n natural numbers)
- $\Sigma(2n-1) = 1 + 3 + 5 + ... + (2n-1) = n^2$

  (Sum of first n odd numbers)
- $\Sigma 2n = 2 + 4 + 6 + ... + 2n = n(n+1)$

  (Sum of first n even numbers)
- $\Sigma n^2 = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$

  (Sum of the squares of first n natural numbers)
- $\Sigma n^3 = 1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$

  (Sum of the cubes of first n natural numbers)

Application in Solving Objective Question :

For solving objective question related to natural numbers we find out the correct alternative by negative examination of this principle. If the given statement is P(n), then by putting n = 1, 2, 3, ... in P(n), we decide the correct answer.

We also use the above formulae established by this principle to find the sum of n terms of a given series. For this we first express T_n as a polynomial in n and then for finding S_n, we put Σ before each term of this polynomial and then use above results of $\Sigma n, \Sigma n^2, \Sigma n^3$ etc.