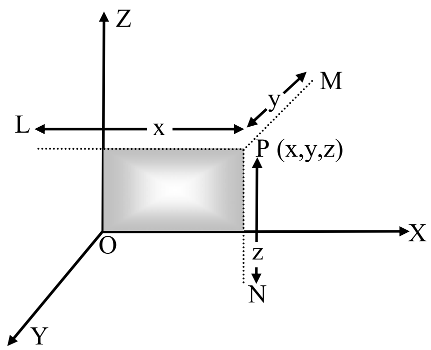
21-3-D GEOMETRY

Coordinates of a point :



x-coordinate = perpendicular distance of P from yz-plane

y-coordinate = perpendicular distance of P from zx-plane

z-coordinate = perpendicular distance of P from xy-plane

Coordinates of a point on the coordinate planes and axes:

yz-plane	:	$x = 0$
zx-plane	:	$y = 0$
xy-plane	:	$z = 0$
x-axis	:	$y = 0, z = 0$
y-axis	:	$y = 0, x = 0$
z-axis	:	$x = 0, y = 0$

Distance between two points :

If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points, then distance between them

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Coordinates of division point :

Coordinates of the point dividing the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $m_1 : m_2$ are

(i) in case of internal division

$$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right)$$

(ii) in case of external division

$$\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}, \frac{m_1z_2 - m_2z_1}{m_1 - m_2}\right)$$

Note: When m_1, m_2 are in opposite sign, then division will be external.

Coordinates of the midpoint:

When division point is the mid-point of PQ, then ration will be 1 : 1; hence coordinates of the mid-point of PQ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Coordinates of the general point :

The coordinates of any point lying on the line joining points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ may be taken as

$$\left(\frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}, \frac{kz_2 + z_1}{k+1}\right)$$

which divides PQ in the ratio $k : 1$. This is called general point on the line PQ.

Division by coordinate planes :

The ratios in which the line segment PQ joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is divided by coordinate planes are as follows :

(i) by yz-plane	:	$-x_1/x_2$ ratio
(ii) by zx-plane	:	$-y_1/y_2$ ratio
(iii) by xy-plane	:	$-z_1/z_2$ ratio

Coordinates of the centroid :

(i) If (x_1, y_1, z_1); (x_2, y_2, z_2) and (x_3, y_3, z_3) are vertices of a triangle then coordinates of its centroid are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

(ii) If (x_r, y_r, z_r); $r = 1, 2, 3, 4$ are vertices of a tetrahedron, then coordinates of its centroid are

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$$

Direction cosines of a line [Dc's] :

The cosines of the angles made by a line with positive direction of coordinate axes are called the direction cosines of that line.

Let α, β, γ be the angles made by a line AB with positive direction of coordinate axes then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of AB which are generally denoted by l, m, n. Hence

$$l = \cos\alpha,\ m = \cos\beta,\ n = \cos\gamma$$

x-axis makes 0º, 90º and 90º angles with three coordinate axes, so its direction cosines are cos 0º, cos 90º, cos 90º i.e. 1, 0, 0. Similarly direction cosines of y-axis and z-axis are 0, 1, 0 and 0, 0, 1 respectively. Hence

dc's of x-axis = 1, 0, 0

dc's of y-axis = 0, 1, 0

dc's of z-axis = 0, 0, 1

Relation between dc's

$\therefore l^2 + m^2 + n^2 = 1$

Direction ratios of a line [DR's] :

Three numbers which are proportional to the direction cosines of a line are called the direction ratios of that line. If a, b, c are such numbers which are proportional to the direction cosines l, m, n of a line then a, b, c are direction ratios of the line. Hence

$$\Rightarrow l = \pm \frac{a}{\sqrt{a^2+b^2+c^2}},$$

$$m = \pm \frac{b}{\sqrt{a^2+b^2+c^2}}, \ n = \pm \frac{c}{\sqrt{a^2+b^2+c^2}}$$

Direction cosines of a line joining two points :

Let ≡ (x_1, y_1, z_1) and Q ≡ (x_2, y_2, z_2); then

(i) dr's of PQ : $(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)$

(ii)dc's of PQ : $\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$

i.e., $\frac{x_2 - x_1}{\sqrt{\Sigma(x_2 - x_1)^2}}, \frac{y_2 - y_1}{\sqrt{\Sigma(x_2 - x_1)^2}}, \frac{z_2 - z_1}{\sqrt{\Sigma(x_2 - x_1)^2}}$

Angle between two lines :

Case I. When dc's of the lines are given

If l_1, m_1, and l_2,m_2 n_2 are dc's of given two lines, then the angle θ between them is given by

- $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$
- $\sin\theta = \sqrt{(\ell_1 m_2 - \ell_2 m_1)^2 + (m_1 n_2 - m_2 n_1)^2 + (n_1 \ell_2 - n_2 \ell_1)^2}$

The value of sin θ can easily be obtained by the following form :

$$\sin\theta = \sqrt{\begin{vmatrix} \ell_1 & m_1 \\ \ell_2 & m_2 \end{vmatrix}^2 + \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} n_1 & \ell_1 \\ n_2 & \ell_2 \end{vmatrix}^2}$$

Case II. When dr's of the lines are given

If a_1, b_1, c_1 and a_2, b_2, c_2 are dr's of given two lines, then the angle θ between them is given by

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\sin\theta = \frac{\sqrt{\Sigma(a_1 b_2 - a_2 b_1)^2}}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Conditions of parallelism and perpendicularity of two lines :

Case I. When dc's of two lines AB and CD, say ℓ_1, m_1,n_1 and ℓ_2, m_2, n_2 are known

AB || CD ⇔ $\ell_1 = \ell_2$, $m_1 = m_2$, $n_1 = n_2$

AB ⊥ CD ⇔ $\ell_1 \ell_2 + m_1 m_2 + n_1 n_2 = 0$.

Case II. When dr's of two lines AB and CD, say : a_1, b_1, c_1 and a_2, b_2, c_2 are known

AB || CD ⇔ $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

AB ⊥ CD ⇔ $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$.

Area of a triangle :

Let $A(x_1, y_1, z_1)$; $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ are vertices of a triangle. Then

dr's of AB = $x_2 - x_1, y_2 - y_1, z_2 - z_1$

= a_1, b_1, c_1 (say)

and AB = $\sqrt{a_1^2 + b_1^2 + c_1^2}$

dr's of BC = $x_3 - x_2, y_3 - y_2, z_3 - z_2$

= a_2, b_2, c_2(say)

and BC = $\sqrt{a_2^2 + b_2^2 + c_2^2}$

Now $\sin B = \frac{\sqrt{\Sigma(b_1 c_2 - b_2 c_1)^2}}{\sqrt{\Sigma a_1^2}\sqrt{\Sigma a_2^2}}$

$= \frac{\sqrt{\Sigma(b_1 c_2 - b_2 c_1)^2}}{AB.BC}$

∴ Area of ΔABC $= \frac{1}{2}$ AB. BC sin B

$= \frac{1}{2}\sqrt{\Sigma(b_1 c_2 - b_2 c_1)^2}$

Projection of a line segment joining two points on a line :

Let PQ be a line segment where P ≡ (x_1, y_1, z_1) and Q ≡ (x_2, y_2, z_2); and AB be a given line with dc's as l, m, n. If P'Q' be the projection of PQ on AB, then

P'Q' = PQ cos θ

where θ is the angle between PQ and AB. On replacing the value of cos θ in this, we shall get the following value of P'Q'.

P'Q' = $l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

Projection of PQ on x-axis : a = $|x_2 - x_1|$

Projection of PQ on y-axis : b = $|y_2 - y_1|$

Projection of PQ on z-axis : c = $|z_2 - z_1|$

Length of line segment PQ = $\sqrt{a^2 + b^2 + c^2}$

* If the given lines are $\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and $\frac{x-\alpha'}{\ell'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'}$, then condition for intersection is

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$$\begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ \ell & m & n \\ \ell & m' & n' \end{vmatrix} = 0$$

Plane containing the above two lines is

$$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} = 0$$

Condition of coplanarity if both the lines are in general form:

Let the lines be

$ax + by + cz + d = 0 = a'x + b'y + c'z + d'$

and $\alpha x + \beta y + \gamma z + \delta = 0 = \alpha' x + \beta' y + \gamma' z + \delta'$

These are coplanar if $\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta' \end{vmatrix} = 0$

Reduction of non-symmetrical form to symmetrical form:

Let equation of the line in non-symmetrical form be'

$a_1x + b_1y + c_1z + d_1 = 0;\ a_2x + b_2y + c_2z + d_2 = 0.$

To find the equation of the line in symmetrical form, we must know (i) its direction ratios (ii) coordinates of any point on it.

- **Direction ratios :** Let ℓ, m, n be the direction ratios of the line. Since the line lies in both the planes, it must be perpendicular to normals of both planes. So

$a_1\ell + b_1m + c_1n = 0;\ a_2\ell + b_2m + c_2n = 0$

From these equations, proportional values of ℓ, m, n can be found by cross-multiplication as

$$\frac{\ell}{b_1c_2 - b_2c_1} = \frac{m}{c_1a_2 - c_2a_1} = \frac{n}{a_1b_2 - a_2b_1}$$

- **Point on the line :** Note that as ℓ, m, n cannot be zero simultaneously, so at least one must be non-zero. Let $a_1b_2 - a_2b_1 \neq 0$, then the line cannot be parallel to xy-plane, so it intersect it. Let it intersect xy-plane in $(x_1, y_1, 0)$. Then

$a_1x_1 + b_1y_1 + d_1 = 0$ and $a_2x_1 + b_2y_1 + d_2 = 0$

Solving these, we get a point on the line. Then its equation becomes

$$\frac{x - x_1}{b_1c_2 - b_2c_1} = \frac{y - y_1}{c_1a_2 - c_2a_1} = \frac{z - 0}{a_1b_2 - a_2b_1}$$

or $$\frac{x - \frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}}{b_1c_2 - b_2c_1} = \frac{y - \frac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}}{c_1a_2 - c_2a_1} = \frac{z - 0}{a_1b_2 - a_2b_1}$$

Note : If $\ell \neq 0$, take a point on yz –plane as $(0, t_1, z_1)$ and if $m \neq 0$, take a point on xz-plane as $(x_1, 0, z_1)$

- **Skew lines :** The straight lines which are not parallel and non-coplanar i.e. non-intersecting are called skew lines.

If $\Delta = \begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} \neq 0$, the lines are skew.

Shortest distance : Suppose the equation of the lines are $\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$

and $\frac{x-\alpha'}{\ell'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'}$. Then

$$\text{S.D.} = \frac{(\alpha-\alpha')(mn'-m'n) + (\beta-\beta')(n\ell'-n'\ell)(\ell m'-\ell' m)}{\sqrt{\Sigma(mn'-m'n)^2}}$$

$$= \begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix}$$

Some results for plane and straight line:

(i) General equation of a plane :

$ax + by + cz + d = 0$

where a, b, c are dr's of a normal to this plane.

(ii) Equation of a straight line :

General form : $\left.\begin{aligned} a_1x + b_1y + c_1z + d_1 = 0 \\ a_2x + b_2y + c_2z + d_2 = 0 \end{aligned}\right\}$

(In fact it is the straight line which is the intersection of two given planes)

Symmetric form : $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

where (x_1, y_1, z_1) is a point on this line and a, b, c are its dr's

(iii) Angle between two planes :

If θ be the angle between planes $a_1x + b_1y\ \ c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, then

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

(In fact angle between two planes is the angle between their normals.)

Further above two planes are

parallel $\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

perpendicular $\Leftrightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$