15-DEFINITE INTEGRAL AND AREAS

Properties 1 :

- If $\int f(x)\,dx = F(x)$, then

  $\int_a^b f(x)\,dx = F(b) - F(a),\ b \ge a$

  Where F(x) is one of the antiderivatives of the function f(x), i.e. $F'(x) = f(x)\ (a \le x \le b)$.

  Remark : When evaluating integrals with the help of the above formula, the students should keep in mind the condition for its legitimate use. This formula is used to compute the definite integral of a function continuous on the interval [a, b] only when the equality $F'(x) = f(x)$ is fulfilled in the whole interval [a, b], where F(x) is antiderivative of the function f(x). In particular, the antiderivative must be a function continuous on the whole interval [a, b]. A discontinuous function used as an antiderivative will lead to wrong result.

- If $F(x) = \int_a^x f(t)\,dt,\ t \ge a$, then $F'(x) = f(x)$

Properties of Definite Integrals :

If $f(x) \ge 0$ on the interval [a, b], then $\int_a^b f(x)\,dx \ge 0$

- $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$
- $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,\ a < c < b$
- $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

or $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

- $\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_a^b f(x)\,dx & \text{if } f(-x) = f(x) \\ 0 & \text{if } f(-x) = -f(x) \end{cases}$
- $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_a^b f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$

- Every continuous function defined on [a, b] is integrable over [a, b].
- Every monotonic function defined on [a, b] is integrable over [a, b]
- If f(x) is a continuous function defined on [a, b], then there exists $c \in (a, b)$such that

  $\int_a^b f(x)\,dx = f(c)\,.\,(b-a)$

  The number $f(c) = \dfrac{1}{(b-a)}\int_a^b f(x)\,dx$ is called the mean value of the function f(x) on the interval [a, b].
- If f is continous on [a, b], then the integral function g defined by $g(x) = \int_a^x f(t)\,dt$ for $x \in [a, b]$ is derivable on [a, b] and $g'(x) = f(x)$ for all $x \in [a, b]$.
- If m and M are the smallest and greatest values of a function f(x) on an interval [a, b], then

  $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$
- If the function $\phi(x)$ and $\psi(x)$ are defined on [a, b] and differentiable at a point $x \in (a, b)$ and f(t) is continuous for $\phi(a) \le t \le \psi(b)$, then

  $\dfrac{d}{dx}\left(\int_{\phi(x)}^{\psi(x)} f(t)\,dt\right) = f(\psi(x))\,\psi'(x) - f(\phi(x))\,\phi'(x)$
- $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$
- If $f^2(x)$ and $g^2(x)$ are integrable on [a, b], then

  $\left|\int_a^b f(x)g(x)\,dx\right| \le \left(\int_a^b f^2(x)\,dx\right)^{1/2}\left(\int_a^b g^2(x)\,dx\right)^{1/2}$
- **Change of variables :** If the function f(x) is continuous on [a, b] and the function $x = \phi(t)$ is continuously differentiable on the interval $[t_1, t_2]$ and $a = \phi(t_1)$, $b = \phi(t_2)$, then

  $\int_a^b f(x)\,dx = \int_{t_1}^{t_2} f(\phi(t))\,\phi'(t)\,dt$
- Let a function $f(x, \alpha)$ be continuous for $a \le x \le b$ and $c \le \alpha \le d$. Then for any $\alpha \in [c, d]$, if

  $I(\alpha) = \int_a^b f(x,\alpha)\,dx$, then $I'(\alpha) = \int_a^b f'(x,\alpha)\,dx$, Where $I'(\alpha)$ is the derivative of $I(\alpha)$ w.r.t. α and $f'(x, \alpha)$ is the derivative of $f(x, \alpha)$ w.r.t. α, keeping x constant.

Integrals with Infinite Limits :

If a function f(x) is continuous for $a \le x < \infty$, then by definition

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx \qquad ....(i)$$

If there exists a finite limit on the right hand side of (i), then the improper integrals is said to be convergent; otherwise it is divergent.

Geometrically, the improper integral (i) for $f(x) > 0$, is the area of the figure bounded by the graph of the function $y = f(x)$, the straight line $x = a$ and the x-axis. Similarly,

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx \text{ and}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$

properties :

- $\int_0^a x f(x)\,dx = \frac{1}{2} a\int_0^a f(x)$ if $f(a - x) = f(x)$ and $\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = \frac{a}{2}$
- $\int_0^{\pi/2} \log\sin x\,dx = \int_0^{\pi/2}\log\cos x\,dx$ $= -\frac{\pi}{2}\log 2 = \frac{\pi}{2}\log\frac{1}{2}$
- $\Gamma(n + 1) = n\,\Gamma(n)$, $\Gamma(1) = 1$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
- If m and n are non-negative integers, then

$$\int_0^{\pi/2}\sin^m x\cos^n x\,dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$

Reduction Formulae of some Define Integrals :

- $\int_0^{\infty} e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$
- $\int_0^{\infty} e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$
- $\int_0^{\infty} e^{-ax} x^n dx = \frac{n!}{a^{n+1}}$
- If $I_n = \int_0^{\pi/2}\sin^n x\,dx$, then

$$I_n = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}.....\frac{2}{3} & \text{(when n is odd)} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}.....\frac{1}{2}\cdot\frac{\pi}{2} & \text{(when n is even)} \end{cases}$$

- If $I_n = \int_0^{\pi/2}\cos^n x\,dx$, then

$$I_m = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}.....\frac{2}{3} & \text{(when n is odd)} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}.....\frac{1}{2}\cdot\frac{\pi}{2} & \text{(when n is even)} \end{cases}$$

Leibnitz's Rule :

If f(x) is continuous and u(x), v(x) are differentiable functions in the interval [a, b], then

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f\{v(x)\}\frac{d}{dx}v(x) - f\{u(x)\frac{d}{dx}u(x)$$

Summation of Series by Integration :

$$\lim_{n\to\infty}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)\cdot\frac{1}{n} = \int_0^1 f(x)\,dx$$

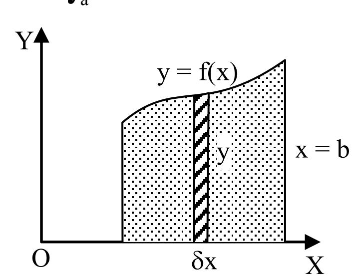
Some Important Results :

- $\sum_{r=0}^{n-1}\sin(\alpha + r\beta) = \frac{\sin\left\{\alpha + \frac{1}{2}(n-1)\beta\right\}\sin\left(\frac{1}{2}n\beta\right)}{\sin\left(\frac{1}{2}\beta\right)}$
- $\sum_{r=0}^{n-1}\cos(\alpha + r\beta) = \frac{\cos\left\{\alpha + \frac{1}{2}(n-1)\beta\right\}\sin\left(\frac{1}{2}n\beta\right)}{\sin\left(\frac{1}{2}\beta\right)}$
- $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - .... = \frac{\pi^2}{12}$
- $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... = \frac{\pi^2}{6}$

Area under Curves :

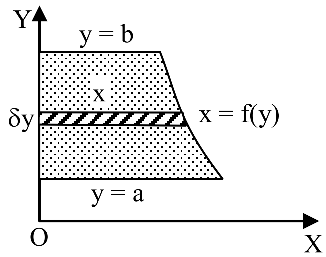
- Area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$

$$= \int_a^b y\,dx = \int_a^b f(x)\,dx$$



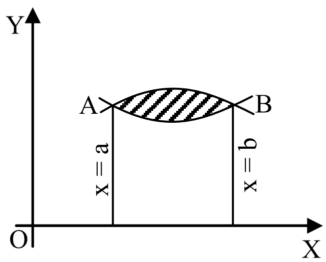
- Area bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = a$, $y = b$

$$= \int_a^b x\, dy = \int_a^b f(y)\, dy$$



- The area of the region bounded by $y_1 = f_1(x)$, $y_2 = f_2(x)$ and the ordinates x = a and x = b is given by

$$= \int_a^b f_2(x)\, dx - \int_a^b f_1(x)\, dx$$

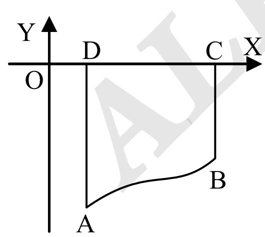


where $f_2(x)$ is y_2 of the upper curve and $f_1(x)$ is y_1 of the lower curve, i.e. the required area

$$= \int_a^b [f_2(x) - f_1(x)]\, dx = \int_a^b (y_2 - y_1)\, dx$$

- $f(x) \le 0$ for all x in $a \le x \le b$, then area bounded by x-axis, the curve y = f(x) and the ordinates x = a, x = b is given by

$$= -\int_a^b f(x)\, dx$$



- If $f(x) \ge 0$ for $a \le x \le c$ and $f(x) \le 0$ for $c \le x \le b$, then area bounded by y = f(x), x-axis and the ordinates x = a, x = b is given by

$$= \int_a^c f(x)\, dx + \int_c^b -f(x)\, dx = \int_a^c f(x)\, dx - \int_c^b f(x)\, dx$$