14-INTEGRATION

Integration :

- If $\frac{d}{dx} f(x) = F(x)$, then $\int F(x)\,dx = f(x) + c$, where c is an arbitrary constant called constant of integration.

1. $\int x^n dx = \frac{x^{n+1}}{n+1} \ (n \neq -1)$

2. $\int \frac{1}{x} dx = \log x$

3. $\int e^x dx = e^x$

4. $\int a^x dx = \frac{a^x}{\log_e a}$

5. $\int \sin x\,dx = -\cos x$

6. $\int \cos x\,dx = \sin x$

7. $\int \sec^2 x\,dx = \tan x$

8. $\int \text{cosec}^2 x\,dx = -\cot x$

9. $\int \sec x \tan x\,dx = \sec x$

10. $\int \text{cosec } x \cot x\,dx = -\text{cosec } x$

11. $\int \sec x\,dx = \log(\sec x + \tan x) = \log \tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$

12. $\int \text{cosec } x\,dx = -\log(\text{cosec } x + \cot x) = \log \tan\left(\frac{x}{2}\right)$

13. $\int \tan x\,dx = -\log \cos x$

14. $\int \cot x\,dx = \log \sin x$

15. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} = -\cos^{-1}\frac{x}{a}$

16. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} = -\frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right)$

17. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} = -\frac{1}{a}\text{cosec}^{-1}\left(\frac{x}{a}\right)$

18. $\int \frac{1}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|$, when $x > a$

19. $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|$, when $x < a$

20. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left\{x + \sqrt{x^2 - a^2}\right\} = \cos h^{-1}\left(\frac{x}{a}\right)$

21. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left\{x + \sqrt{x^2 + a^2}\right\} = \sin h^{-1}\left(\frac{x}{a}\right)$

22. $\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)$

23. $\int \sqrt{x^2 - a^2}\,dx = \frac{1}{2}x\sqrt{x^2 - a^2} - \frac{1}{2}a^2\log\left\{x + \sqrt{x^2 - a^2}\right\}$

24. $\int \sqrt{x^2 + a^2}\,dx = \frac{1}{2}x\sqrt{x^2 + a^2} + \frac{1}{2}a^2\log\left\{x + \sqrt{x^2 + a^2}\right\}$

25. $\int \frac{f'(x)}{f(x)} dx = \log f(x)$

26. $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)}$

Integration by Decomposition into Sum :

1. **Trigonometrical transformations :** For the integrations of the trigonometrical products such as $\sin^2 x$, $\cos^2 x$, $\sin^3 x$, $\cos^3 x$, sin ax cos bx, etc., they are expressed as the sum or difference of the sines and cosines of multiples of angles.

2. **Partial fractions :** If the given function is in the form of fractions of two polynomials, then for its integration, decompose it into partial fractions (if possible).

Integration of some special integrals :

(i) $\int \frac{dx}{ax^2 + bx + c}$

This may be reduced to one of the forms of the above formulae (16), (18) or (19).

(ii) $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

This can be reduced to one of the forms of the above formulae (15), (20) or (21).

(iii) $\int \sqrt{ax^2+bx+c}\,dx$

This can be reduced to one of the forms of the above formulae (22), (23) or (24).

(iv) $\int \frac{(px+q)dx}{ax^2+bx+c}$, $\int \frac{(px+q)dx}{\sqrt{ax^2+bx+c}}$

For the evaluation of any of these integrals, put $px + q = A$ {differentiation of $(ax^2 + bx + c)$} + B

Find A and B by comparing the coefficients of like powers of x on the two sides.

- **1.** If k is a constant, then

$$\int k\,dx = kx \text{ and } \int k f(x)dx = k\int f(x)dx$$

2. $\int \{f_1(x) \pm f_2(x)\}\,dx = \int f_1(x)dx \pm \int f_2(x)dx$

Some Proper Substitutions :

1. $\int f(ax+b)\,dx$, $ax + b = t$

2. $\int f(ax^n+b)x^{n-1}dx$, $ax^n + b = t$

3. $\int f\{\phi(x)\}\,\phi'(x)\,dx$, $\phi(x) = t$

4. $\int \frac{f'(x)}{f(x)}dx$, $f(x) = t$

5. $\int \sqrt{a^2-x^2}\,dx$, $x = a\sin\theta$ or $a\cos\theta$

6. $\int \sqrt{a^2+x^2}\,dx$, $x = a\tan\theta$

7. $\int \sqrt{\frac{a^2-x^2}{a^2+x^2}}\,dx$, $x^2 = a^2\cos 2\theta$

8. $\int \sqrt{a \pm x}\,dx$, $a \pm x = t^2$

9. $\int \sqrt{\frac{a-x}{a+x}}\,dx$, $x = a\cos 2\theta$

10. $\int \sqrt{2ax-x^2}\,dx$, $x = a(1-\cos\theta)$

11. $\int \sqrt{x^2-a^2}\,dx$, $x = a\sec\theta$

Substitution for Some irrational Functions :

1. $\int \frac{dx}{(px+q)\sqrt{ax+b}}$, $ax + b = t^2$

2. $\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$, $px + q = \frac{1}{t}$

3. $\int \frac{dx}{(px^2+qx+r)\sqrt{ax+b}}$, $ax + b = t^2$

4. $\int \frac{dx}{(px^2+r)\sqrt{ax^2+c}}$, at first $x = \frac{1}{t}$ and then $a + ct^2 = z^2$

Some Important Integrals :

1. To evaluate $\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$, $\int \sqrt{\left(\frac{x-\alpha}{\beta-x}\right)}\,dx$, $\int \sqrt{(x-\alpha)(\beta-x)}\,dx$. Put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

2. To evaluate $\int \frac{dx}{a+b\cos x}$, $\int \frac{dx}{a+b\sin x}$, $\int \frac{dx}{a+b\cos x+c\sin x}$

Replace $\sin x = \frac{\left(2\tan\frac{x}{2}\right)}{\left(1+\tan^2\frac{x}{2}\right)}$ and $\cos x = \frac{\left(1-\tan^2\frac{x}{2}\right)}{\left(1+\tan^2\frac{x}{2}\right)}$

Then put $\tan\frac{x}{2} = t$.

3. To evaluate $\int \frac{p\cos x+q\sin x}{a+b\cos x+c\sin x}dx$

Put $p\cos x + q\sin x = A(a + b\cos x + c\sin x)$
$+ B.$ diff. of $(a + b\cos x + c\sin x) + C$

A, B and C can be calculated by equating the coefficients of cos x, sin x and the constant terms.

4. To evaluate $\int \frac{dx}{a\cos^2 x+2b\sin x\cos x+c\sin^2 x}$, $\int \frac{dx}{a\cos^2 x+b}$, $\int \frac{dx}{a+b\sin^2 x}$

In the above type of questions divide N^r and D^r by $\cos^2 x$. The numerator will become $\sec^2 x$ and in the denominator we will have a quadratic equation in tan x (change $\sec^2 x$ into $1 + \tan^2 x$).

Putting tan x = t the question will reduce to the form $\int \frac{dt}{at^2+bt+c}$

5. **Integration of rational function of the given form**

(i) $\int \frac{x^2+a^2}{x^4+kx^2+a^4}dx$, (ii) $\int \frac{x^2-a^2}{x^4+kx^2+a^4}dx$, where k is a constant, positive, negative or zero.

These integrals can be obtained by dividing numerator and denominator by x^2, then putting $x - \frac{a^2}{x} = t$ and $x + \frac{a^2}{x} = t$ respectively.

Integration of Product of Two Functions :

1. $\int f_1(x)\,f_2(x)\,dx = f_1(x)\int f_2(x)\,dx - \int \left[\left(f_1'(x)\int f_2(x)dx\right)\right]dx$

Proper choice of the first and second functions : Integration with the help of the above rule is called integration by parts, In the above rule, there are two terms on R.H.S. and in both the terms integral of the second function is involve. Therefore in the product of two functions if one of the two functions is not directly integrable (e.g. log x, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc.) we take it as the first function and the remaining function is taken as the second function. If there is no other function, then unity is taken as the second function. If in the integral both the functions are easily integrable, then the first function is chosen in such a way that the derivative of the function is a simple functions and the function thus obtained under the integral sign is easily integrable than the original function.

2. $\int e^{ax}\sin(bx+c)\,dx$

$$= \frac{e^{ax}}{a^2+b^2}[a\sin(bx+c) - b\cos(bx+c)]$$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left[bx+c-\tan^{-1}\frac{b}{a}\right]$$

3. $\int e^{ax}\cos(bx+c)\,dx$

$$= \frac{e^{ax}}{a^2+b^2}[a\cos(bx+c) + b\sin(bx+c)]$$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left[bx+c-\tan^{-1}\frac{b}{a}\right]$$

4. $\int e^{kx}\{kf(x) + f'(x)\}\,dx = e^{kx}f(x)$

5. $\int \log_e x = x(\log_e x - 1) = x\log_e\left(\frac{x}{e}\right)$

Integration of Trigonometric Functions :

1. To evaluate the integrals of the form

$I = \int \sin^m x\cos^n x\,dx$, where m and n are rational numbers.

(i) Substitute sin x = t, if n is odd;

(ii) Substitute cos x = t, if m is odd;

(iii) Substitute tan x = t, if m + n is a negative even integer; and

(iv) Substitute cot x = t, if $\frac{1}{2}(m+1) + \frac{1}{2}(n-1)$ is an integer.

2. Integrals of the form $\int R(\sin x, \cos x)\,dx$, where R is a rational function of sin x and cos x, are transformed into integrals of a rational function by the substitution $\tan\frac{x}{2} = t$, where $-\pi < x < \pi$. This is the so called universal substitution. Sometimes it is more convenient to make the substitution $\cot\frac{x}{2} = t$ for $0 < x < 2\pi$.

The above substitution enables us to integrate any function of the form R (sin x, cos x). However, in practice, it sometimes leads to extremely complex rational functions. In some cases, the integral can be simplified by –

(i) Substituting sin x = t, if the integral is of the form $\int R(\sin x)\cos x\,dx$.

(ii) Substituting cos x = t, if the integral is of the form $\int R(\cos x)\sin x\,dx$.

(iii) Substituting tan x = t, i.e. $dx = \frac{dt}{1+t^2}$, if the integral is dependent only on tan x.

Some Useful Integrals :

1. (When a > b) $\int \frac{dx}{a+b\cos x}$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right] + c$$

2. (When a < b) $\int \frac{dx}{a+b\cos x}$

$$= -\frac{1}{\sqrt{b^2-a^2}}\log\left|\frac{\sqrt{b-a}\tan\frac{x}{a} - \sqrt{a+b}}{\sqrt{b-a}\tan\frac{x}{a} + \sqrt{a+b}}\right|$$

3. (when a = b) $\int \frac{dx}{a+b\cos x} = \frac{1}{a}\tan\frac{x}{2} + c$

4. (When a > b) $\int \frac{dx}{a+b\sin x}$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left\{\frac{a\tan\left(\frac{x}{2}\right)+b}{\sqrt{a^2-b^2}}\right\} + c$$

5. (When a < b) $\int \frac{dx}{a+b\sin x}$

$$= \frac{1}{\sqrt{b^2-a^2}}\log\left|\frac{a\tan\left(\frac{x}{2}\right)+b-\sqrt{b^2-a^2}}{a\tan\left(\frac{x}{2}\right)+b+\sqrt{b^2-a^2}}\right| + c$$

6. (When a = b) $\int \frac{dx}{a+b\sin x} = \frac{1}{a}[\tan x - \sec x] + c$