10-LIMITS,CONTINUITY AND DIFFERENTIABILITY

Limits :

Theorems of Limits :

If f(x) and g(x) are two functions, then

(i) $\lim_{x\to a}[f(x)\pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

(ii) $\lim_{x\to a}[f(x).g(x)] = \lim_{x\to a} f(x) \,.\, \lim_{x\to a} g(x)$

(iii) $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

(iv) $\lim_{x\to a}[kf(x)] = k \lim_{x\to a} f(x)$, where k is constant.

(v) $\lim_{x\to a}\sqrt{f(x)} = \sqrt{\lim_{x\to a} f(x)}$

(vi) $\lim_{x\to a}|f(x)|^{p/q} = \left(\lim_{x\to a} f(x)\right)^{p/q}$, where p and q are integers.

Some important expansions :

(i) $\sin x = \left\{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...\right\}$

(ii) $\cos x = \left\{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...\right\}$

(iii) $\sin h\, x = \left\{x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...\infty\right\}$

(iv) $\cos h\, x = \left\{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...\infty\right\}$

(v) $\tan x = \left\{x + \frac{x^3}{3} + \frac{2x^5}{15} + ...\right\}$

(vi) $\log(1 + x) = \left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\right\}$

(vii) $e^x = \left\{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right\}$

(viii) $a^x = \left\{1 + x\log a + \frac{x^2}{2!}(\log a)^2 + ...\right\}$

(ix) $(1 - x)^{-1} = \{1 + x + x^2 + x^3 + ...\}$

(x) $\sin^{-1}x = \left\{x + \frac{1}{2}.\frac{x^3}{3} + \frac{1}{2}.\frac{3}{4}.\frac{x^5}{5} + \frac{1}{2}.\frac{3}{4}.\frac{5}{6}.\frac{x^7}{7} + ...\right\}$

(xi) $\tan^{-1}x = \left\{x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - .....\right\}$

Some important Limits :

(i) $\lim_{x\to 0} \sin x = 0$

(ii) $\lim_{x\to 0} \cos x = 1$

(iii) $\lim_{x\to 0} \frac{\sin x}{x} = 1 = \lim_{x\to 0} \frac{x}{\sin x}$

(iv) $\lim_{x\to 0} \frac{\tan x}{x} = 1 = \lim_{x\to 0} \frac{x}{\tan x}$

(v) $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

(vi) $\lim_{x\to 0} e^x = 1$

(vii) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

(viii) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$

(ix) $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

(x) $\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e = \lim_{x\to-\infty}\left(1 + \frac{1}{x}\right)^x$

(xi) $\lim_{x\to 0}(1 + x)^{1/x} = e$

(xii) $\lim_{x\to\infty}\left(1 + \frac{a}{x}\right)^x = e^a$

(xiii) $\lim_{x\to\infty} a^n = \begin{cases} \infty, & \text{if } a > 1 \\ 0, & \text{if } a < 1 \end{cases}$

i.e. $a^\infty = \infty$, if a > 1 and $a^\infty = 0$, if a < 1

(xiv) $\lim_{x\to 0} \frac{(1+x)^n - 1}{x} = n$

(xv) $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1 = \lim_{x\to 0} \frac{\tan^{-1} x}{x}$

(xvi) $\lim_{x\to a} \sin^{-1} x = \sin^{-1}a, |a| \le 1$

(xvii) $\lim_{x\to a} \cos^{-1} x = \cos^{-1}a, |a| \le 1$

(xviii) $\lim_{x\to a} \tan^{-1} x = \tan^{-1}a, -\infty < a < \infty$

(xix) $\lim_{x\to e} \log_e x = 1$

(xx) $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

Let $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = m$, then

(xxi) $\lim_{x\to a} (f(x))^{g(x)} = \ell^m$

(xxii) If $f(x) \le g(x)$ for every x in the deleted neighbourhood (nbd) of a, then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$.

(xxiii) If $f(x) \le g(x) \le h(x)$ for every x in the deleted nbd of a and $\lim_{x\to a} f(x) = \ell = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = \ell$.

(xxiv) $\lim_{x\to a} fog(x) = f\left(\lim_{x\to a} g(x)\right) = f(m)$

In particular (a) $\lim_{x\to a} \log f(x) = \log\left(\lim_{x\to a} f(x)\right) = \log \ell$

(b) $\lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a} f(x)} = e^{\ell}$

(xxv) If $\lim_{x\to a} f(x) = +\infty$ or $-\infty$, then $\lim_{x\to a} \frac{1}{f(x)} = 0$.

Evaluation of Limits (Working Rules) :

By factorisation : To evaluate $\lim_{x\to a} \frac{\phi(x)}{\psi(x)}$, factorise both $\phi(x)$ and $\psi(x)$, if possible, then cancel the common factor involving a from the numerator and the denominator. In the last obtain the limit by substituting a for x.

Evaluation by substitution : To evaluate $\lim_{x\to a} f(x)$, put x = a + h and simplify the numerator and denominator, then cancel the common factor involving h in the numerator and denominator. In the last obtain the limit by substituting h = 0.

By L – Hospital's rule : Apply L-Hospital's rule to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f^n(x)}{g^n(x)}$$

By using expansion formulae : The expansion formulae can also be used with advantage in simplification and evaluation of limits.

By rationalisation : In case if numerator or denominator (or both) are irrational functions, rationalisation of numerator or denominator (or both) helps to obtain the limit of the function.

Continuity :

f(x) is continuous at x = a if $\lim_{x\to a} f(x)$ exists and is equal to f(a) i.e. if $\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x)$.

Discontinuous functions : A function f is said to be discontinuous at a point a of its domain D if is not continuous there at. The point a is then called a point of discontinuity of the function. The discontinuity may arise due to any of the following situations:

(a) $\lim_{x\to a+} f(x)$ or $\lim_{x\to a-} f(x)$ of both may not exist.

(b) $\lim_{x\to a+} f(x)$ as well as $\lim_{x\to a-} f(x)$ may exist but are unequal.

(c) $\lim_{x\to a+} f(x)$ as well as $\lim_{x\to a-} f(x)$ both may exist but either of the two or both may not be equal to f(a).

We classify the point of discontinuity according to various situations discussed above.

Removable discontinuity : A function f is said to have removable discontinuity at x = a if

$\lim_{x\to a-} f(x) = \lim_{x\to a+} f(x)$ but their common value is not equal to f(a). Such a discontinuity can be removed by assigning a suitable value to the function f at x = a.

Discontinuity of the first kind : A function f is said to have a discontinuity of the first kind at x = a if $\lim_{x\to a-} f(x)$ and $\lim_{x\to a+} f(x)$ both exist but are not equal.

f is said to have a discontinuity of the first kind from the left at x = a if $\lim_{x\to a-} f(x)$ exists but not equal to f(a). Discontinuity of the first kind from the right is similarly defined.

Discontinuity of second kind : A function f is said to have a discontinuity of the second kind at x = a if neither $\lim_{x\to a-} f(x)$ nor $\lim_{x\to a+} f(x)$ exists.

f if said to have discontinuity of the second kind from the left at x = a if $\lim_{x\to a-} f(x)$ does not exist.

Similarly, if $\lim_{x\to a+} f(x)$ does not exist, then f is said to have discontinuity of the second kind from the right at x = a.

Differentiability :

f(x) is said to be differentiable at x = a if R′ = L′

i.e. $\underset{h\to 0}{\text{Lt}} \frac{f(a+h)-f(a)}{h} = \underset{h\to 0}{\text{Lt}} \frac{f(a-h)-f(a)}{-h}$

Note : We discuss R, L or R′, L′ at x = a when the function is defined differently for x > a or x < a and at x = a.