1-COMPLEX NUMBERS

- $\sqrt{-1}$ is denoted by 'i' and is pronounced as 'iota'.
  $i = \sqrt{-1} \Rightarrow i^2 = -1, i^3 = -i, i^4 = 1.$
- If a, b $\in$ R and i = $\sqrt{-1}$ then a + ib is called a complex number. The complex number a + ib is also denoted by the ordered pair (a, b)
- If z = a + ib is a complex number, then :
  (i) a is called the real part of z and we write
  Re (z) = a.
  (ii) b is called the imaginary part of z and we write
  Im (z) = b
- Two complex numbers z_1 and z_2 are said to be equal complex numbers if Re (z_1) = Re (z_2) and Im (z_1) = Im (z_2).
- If z = x + iy is a non zero complex number, then 1/z is called the multiplicative inverse of z.
- If x + iy is a complex number, then the complex number x – iy is called the conjugate of the complex number x + iy and we write $\overline{x+iy} = x - iy$.
- **Algebra of Complex Numbers**
  (i) **Addition :** (a + ib) + (c + id) = (a + c) + i(b + d)
  (ii) **Subtraction :**
  (a + ib) – (c + id) = (a – c) + i(b – d)
  (iii) Multiplication :
  (a + ib) + (c + id) = (ac – bd) + i(ab + bc)
  (iv) **Division by a non-zero complex number :**
  $$\frac{a+ib}{c+id} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}, (c + id) \neq 0$$
- **Properties** : If z_1, z_2 are complex numbers, then
  (i) $\overline{(\overline{z}_1)} = z_1$
  (ii) $z + \overline{z} = 2 \text{ Re } (z)$
  (iii) $z - \overline{z} = 2i \text{ Im } (z)$
  (iv) $z = \overline{z}$ iff z is purely real
  (v) $z = \overline{z}$ iff z is purely imaginary
  (vi) $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$
  (vii) $\overline{z_1 - z_2} = \overline{z}_1 - \overline{z}_2$
  (viii) $\overline{z_1 . z_2} = \overline{z}_1 . \overline{z}_2$
  (ix) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}$ provided $z_2 \neq 0$
- If x + iy is a complex number, then the non-negative ral number $\sqrt{x^2+y^2}$ is called the modulus of the complex number x + iy and write
  $| x + iy| = \sqrt{x^2+y^2}$
  Properties : If z_1, z_2 are complex numbers, then
  (i) $| z_1 | = 0$ iff $z_1 = 0$
  (ii) $| z_1 | = | \overline{z}_1 | = | -z_1 |$
  (iii) $-| z_1 | \leq \text{Re } (z_1) \leq | z_1 |$
  (iv) $-| z_1 | \leq \text{Im } (z_1) \leq | z_1 |$
  (v) $| z_1 \overline{z}_1 | = | z_1 |^2$
  (vi) $| z_1 + z_2 | \leq | z_1 | + | z_2 |$
  (vii) $| z_1 - z_2 | \geq | z_1 | - | z_2 |$
  (viii) $| z_1 z_2 | = | z_1 | | z_2 |$
  (ix) $\left|\frac{z_1}{z_2}\right| = \frac{| z_1 |}{| z_2 |}$, provided $z_2 \neq 0$
  (x) $| z_1 + z_2 |^2 = | z_1 |^2 + | z_2 |^2 + 2 \text{ Re } (z_1 \overline{z}_2)$
  (xi) $| z_1 - z_2 |^2 = | z_1 |^2 + | z_2 |^2 - 2 \text{ Re } (z_1 \overline{z}_2)$
  (xi) $| z_1 + z_2 |^2 + | z_1 - z_2 |^2 = 2 [| z_1 |^2 + | z_2 |^2]$.
- De Moivre's Theorem
  (i) If n is any integer (positive or negative), then
  $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$
  (ii) If n is a rational number, then the value or one of the values of $(\cos \theta + i \sin \theta)^n$ is $\cos n\theta + i \sin n\theta$
- Euler's Formula
  $e^{i\theta} = \cos \theta + i \sin \theta$ and $e^{-i\theta} = \cos \theta - i \sin \theta$
- Square root of complex number
  Square root of z = a + ib are given by
  $\pm\left[\sqrt{\left(\frac{|z|+a}{2}\right)} + i\sqrt{\left(\frac{|z|-a}{2}\right)}\right]$ for b > 0 and
  $\pm\left[\sqrt{\left(\frac{|z|+a}{2}\right)} - i\sqrt{\left(\frac{|z|-a}{2}\right)}\right]$ for b < 0.

- If $\omega = \frac{-1+i\sqrt{3}}{2}$, then the cube roots of unity are 1, ω and ω^2. We have:

  (i) $1 + \omega + \omega^2 = 0$ (ii) $\omega^3 = 1$

- Let $z = x + iy$ be any complex number.

  Let $z = r(\cos\theta + i\sin\theta)$ where $r > 0$.

  $\therefore x = r\cos\theta$ and $y = r\sin\theta$

  $\therefore x^2 + y^2 = r^2$

  $\Rightarrow \quad r = \sqrt{x^2+y^2} \quad (\because r > 0)$

  $\therefore \cos\theta = \frac{x}{\sqrt{x^2+y^2}}$ and $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$

  The value of θ is found by solving these equations. θ is called the argument (or amplitude) of z.

  If $-p < \theta \le \pi$, then θ is called the principal argument of z.

- Identification of θ –

x	y	arg(z)	Interval of θ
+	+	θ	$\left(0 < \theta < \frac{\pi}{2}\right)$
+	–	$-\theta$	$\left(\frac{-\pi}{2} < \theta < 0\right)$
–	+	$(\pi - \theta)$	$\left(\frac{\pi}{2} < \theta < \pi\right)$
–	–	$-(\pi - \theta)$	$\left(-\pi < \theta < \frac{-\pi}{2}\right)$

- If z_1 and z_2 are two complex numbers then

  (i) $|z_1 - z_2|$ is the distance between the points with affixes z_1 and z_2.

  (ii) $\frac{mz_2+nz_1}{m+n}$ is the affix of the point dividing the line joining the points with affixes z_1 and z_2 in the ratio m : n internally.

  (iii) $\frac{mz_2-nz_1}{m-n}$ is the affix of the point dividing the line joining the points with affixes z_1 and z_2 in the ratio m : n externally where $m \ne n$.

  (iv) If z_1, z_2, z_3 are the affixes of the vertices of a triangle then the affix of its centroid is $\frac{z_1+z_2+z_3}{3}$.

  (v) $z = tz_1 + (1 - t)z_2$ is the equation of the line joining points with affixes z_1 and z_2. Here 't' is a parameter.

  (vi) $\frac{z-z_1}{z_2-z_1} = \frac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1}$ is the equation of the line joining points with affixes z_1 and z_2.

- Three points with affixes z_1, z_2, z_3 are collinear if

  $$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0.$$

- The general equation of a straight line is

  $\bar{a}z + a\bar{z} + b = 0$, where b is any real number.

- (i) $|z - z_1| < r$ represents the circle with centre z_1 and radius r.

  (ii) $|z - z_1| < r$ represents the interior of the circle with centre z_1 and radius r.

- $\left|\frac{z-z_1}{z-z_1}\right| = k$ represents a circle line which is the perpendicular bisector of the line segment joining points with affixes z_1 and z_2.

- $(z - z_1)\ (\bar{z}-\bar{z}_2) + (\bar{z}-\bar{z}_1) + (z - z_2) = 0$ represents the circle with line joining points with affixes z_1 and z_2 as a diameter.

- $|z - z_1| + |z - z_2| = 2k$, $k \in R^+$ represents the ellipse with foci at points with affixes z_1 and z_2.

- If z_1, z_2, z_3 be the affixes of the points A, B, C respectively, then the angle between AB and AC is given by arg $\left(\frac{z_3-z_1}{z_2-z_1}\right)$.

- If z_1, z_2, z_3, z_4 are the affixes of the points A, B, C, D respectively, then the angle between AB and CD is given by arg $\left(\frac{z_2-z_1}{z_4-z_3}\right)$.

- nth roots of a complex number

  Let $z = r(\cos\theta + i\sin\theta)$, $r > 0$ be any complex number. nth root o $z = z^{1/n}$

  $$= r^{1/n}\left(\cos\frac{2k\pi+\theta}{n} + i\sin\frac{2k\pi+\theta}{n}\right),$$

  where k = 0, 1, 2, ………, n – 1.

  There are n distinct values and sum of all these values is 0.

- Logarithm of a complex number

  Let $z = re^{i\theta}$ be any complex number.

  Then $\log z = \log re^{i\theta} = \log r + \log e^{i\theta}$

  $= \log r + i\theta \log e = \log r + i\theta$.

  $\therefore \quad \log z = \log|z| + i\ \mathrm{amp}(z)$.