QUARDRATIC EQUATIONS

General quadratic equation :

An equation of the form

$ax^2 + bx + c = 0$ ...(1)

where $a \neq 0$, is called a quadratic equation, in the real or complex coefficients a, b and c.

Roots of a quadratic equation :

The values of x, (say $x = \alpha, \beta$) which satisfy the quadratic equation (1) are called the roots of the equation and they are given by

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}\ ;\ \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Discriminant of a quadratic equation :

The quantity $D \equiv b^2 - 4ac$, is known as the discriminant of the equation.

Nature of the Roots :

In the equations $ax^2 + bx + c = 0$, let us suppose that a, b, c are real and $a \neq 0$. The following is true about the nature of its roots-

(i) The equation has real and distinct roots if and only if $D \equiv b^2 - 4ac > 0$.

(ii) The equation has real and coincident (equal) roots if and only if $D \equiv b^2 - 4ac = 0$.

(iii) The equation has complex roots of the form $\alpha \pm i\beta$, $\alpha \neq 0$, $\beta \neq 0 \in R$, if and only if $D \equiv b^2 - 4ac < 0$.

(iv) The equation has rational roots if and only if a, b, $c \in Q$ (the set of rational numbers) and $D \equiv b^2 - 4ac$ is a perfect square (of a rational number).

(v) The equation has (unequal) irrational (surd form) roots if and only if $D \equiv b^2 - 4ac > 0$ and not a perfect square even if a, b and c are rational. In this case if $p + \sqrt{q}$, p, q rational, is an irrational root, then $P - \sqrt{q}$ is also a root (a, b, c being rational).

(vi) $\alpha + i\beta$ ($\beta \neq 0$ and $\alpha, \beta \in R$) is a root if and only if its conjugate $\alpha - i\beta$ is a root, that is complex roots occur in pairs in a quadratic equation. In case the equations is satisfied by more than two complex numbers, then it reduces to an identitiy.

$0.x^2 + 0.x + 0 = 0$, i.e. $a = 0 = b = c$.

Relation between Roots and Coefficients :

If α, β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then the sum and product of the roots is

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

Hence the quadratic equation whose roots are α and β is given by

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$ or $(x - \alpha)(x - \beta) = 0$

Condition that the two quadratic equations have a common root :

Let α be a common root of two quadratic equations

$a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$

where $a_1 \neq 0$, $a_2 \neq 0$ and $a_1b_2 - a_2b_1 \neq 0$.

Then $a_1\alpha^2 + b_1\alpha + c_1 = 0$ and $a_2\alpha^2 + b_2\alpha + c_2 = 0$

which gives (by cross multipication),

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

Thus eliminating α, the condition for a common root is given by

$(c_1a_2 - c_2a_1)^2 = (b_1c_2 - b_2c_1)(a_1b_2 - a_2 b_1)$ ...(2)

Condition that the two quadratic equations have both the roots common :

The two quadratic equation will have the same roots if and only if their coefficients are proportional, i.e.

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Descarte's rule of signs :

The maximum number of positive of a polynomial f(x) is the number of changes of signs in f(x) and the maximum number of negative roots of f(x) is the number of changes of signs in f(–x).

Position of roots :

If f(x) = 0 is an equation and a, b are two real numbers such that f(a) f(b) < 0, then the equation f(x) = 0 has at least one real root or an odd number of real roots between a and b. In case f(a) and f(b) are of the same sign, then either no real root or an even number of real roots of f(x) = 0 lie between a and b.

The quadratic expression :

(A) Let $f(x) \equiv ax^2 + bx + c$, $a, b, c \in R$, $a > d$ be a quadratic expression. Since,

$$f(x) = a\left\{\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right)\right\} \qquad ...(3)$$

The following is true from equation (3)

(i) $f(x) > 0$ (< 0) for all values of $x \in R$ if and only if $a > 0$ (< 0) and $D \equiv b^2 - 4ac < 0$.

(ii) $f(x) \geq 0$ (≤ 0) if and only if $a > 0$ (< 0) and $D \equiv b^2 - 4ac = 0$.

In this case ($D = 0$), $f(x) = 0$ if and only if $x = -\frac{b}{2a}$

(iii) If $D \equiv b^2 - 4ac > 0$ and $a > 0$ (< 0), then

$$f(x) = \begin{cases} < 0 (> 0), & \text{for x lying between the roots of } f(x) = 0 \\ > 0 (< 0), & \text{for x not lying between the roots of } f(x) = 0 \\ = 0, & \text{for x = each of the roots of } f(x) = 0 \end{cases}$$

(iv) If $a > 0$, (< 0), then $f(x)$ has a minimum (maximum) value at $x = -\frac{b}{2a}$ and this value is given by

$$[f(x)]_{\min\,(\max)} = \frac{4ac - b^2}{4a}$$

(B) **The sign of the expression :**

(i) The value of expression $(x - a)(x - b)$; ($a < b$) is positive if $x < a$ or $x > b$, in other words x does not lie between a and b.

(ii) The expression $(x - a)(x - b)$; ($a < b$) is negative if $a < x < b$ i.e. if x lies between a and b.

Some important results :

- If $f(\alpha) = 0$ and $f'(\alpha) = 0$, then α is a repeated root of the quadratic equation $f(x) = 0$ and $f(x) = a(x - \alpha)^2$. In fact $\alpha = -\frac{b}{2a}$.
- Imaginary and irrational roots occur in conjugate pairs (when a, b, c $\in$ R or a, b, c being rational) i.e., if $-3 + 2i$ or $5 - 2\sqrt{7}$ is a root then $-3 - 2i$ or $5 + 2\sqrt{7}$ will also be a root.
- For the quadratic equation $ax^2 + bx + c = 0$

  (i) One root will be reciprocal of the other if $a = c$.

  (ii) One root is zero if $c = 0$

  (iii) Roots are equal in magnitude but opposite in sign if $b = 0$.

  (iv) Both roots are zero if $b = c = 0$.

  (v) Roots are positive if a and c are of the same sign and b is of the opposite sign.

  (vi) Roots are of opposite sign if a and c are of opposite sign.

  (vii) Roots are negative if a, b, c are of the same sign.
- If the ratio of roots of the quadratic equation $ax^2 + bx + c = 0$ be p : q, then $pqb^2 = (p + q)^2ac$.
- If one root of the quadratic equation $ax^2 + bx + c = 0$ be p : q, then $pqb^2 = (p + q)^2ac$.
- If one root of the quadratic equation $ax^2 + bx + c= 0$ is equal to the n^{th} power of the other, then

  $$(ac^n)^{\frac{1}{n+1}} + (a^nc)^{\frac{1}{n+1}} + b = 0$$
- If one roots of the equation $ax^2 + bx + c = 0$ be n times the other root, then $nb^2 = ac(n + 1)^2$.
- If the roots of the equation $ax^2 + bx + c = 0$ are of the form $\frac{k+1}{k}$ and $\frac{k+2}{k+1}$, then $(a + b + c)^2 = b^2 - 4ac$.
- If the roots of $ax^2 + bx + c = 0$ are α, β, then the roots of $cx^2 + bx + a = 0$ will be $\frac{1}{\alpha}, \frac{1}{\beta}$.
- The roots of the equation $ax^2 + bx + c = 0$ are reciprocal to $a'x^2 + b'x + c' = 0$ if $(cc' - aa')^2 = (ba' - cb')(ab' - bc')$.
- Let $f(x) = ax^2 + bx + c$, where $a > 0$ .Then

  (i) Conditions for both the roots of $f(x) = 0$ to be greater than a given number K are $b^2 - 4ac \geq 0$; $f(K) > 0$; $\frac{-b}{2a} > K$.

  (ii) The number K lies between the roots of $f(x) = 0$ if $f(K) < 0$.'

  (iii) Condition for exactly one root of $f(x) = 0$ to lie between d and e is $f(d)\,f(e) < 0$.