MATRICES AND DETERMINANTS

Matrices :

- An m × n matrix is a rectangular array of mn numbers (real or complex) arranged in an ordered set of m horizontal lines called rows and n vertical lines called columns enclosed in parentheses. An m × n matrix A is usually written as :

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & & & \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

  Where $1 \le i \le m$ and $1 \le j \le n$
  and is written in compact form as $A = [a_{ij}]_{m\times n}$

- A matrix $A = [a_{ij}]_{m \times n}$ is called
  (i) a rectangular matrix if $m \ne n$
  (ii) a square matrix if m = n
  (iii) a row matrix or row vector if m = 1
  (iv) a column matrix or column vector if n = 1
  (v) a null matrix if $a_{ij} = 0$ for all i, j and is denoted by $O_{m\times n}$
  (vi) a diagonal matrix if $a_{ij} = 0$ for $i \ne j$
  (vii) a scalar matrix if $a_{ij} = 0$ for $i \ne j$ and all diagonal elements a_{ii} are equal

- Two matrices can be added only when thye are of same order. If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, then sum of A and B is denoted by A + B and is a matrix $[a_{ij} + b_{ij}]_{m \times n}$

- The product of two matrices A and B, written as AB, is defined in this very order of matrices if number of columns of A (pre factor) is equal to the number of rows of B (post factor). If AB is defined , we say that A and B are conformable for multiplication in the order AB.
  If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$, then their product AB is a matrix $C = [c_{ij}]_{m \times p}$ where
  C_{ij} = sum of the products of elements of ith row of A with the corresponding elements of jth column of B.

- Types of matrices :
  (i) Idempotent if $A^2 = A$
  (ii) Periodic if $A^{k+1} = A$ for some positive integer k. The least value of k is called the period of A.
  (iii) Nilpotent if $A^k = O$ when k is a positive integer. Least value of k is called the index of the nilpotent matrix.
  (iv) Involutary if $A^2 = I$.

- The matrix obtained from a matrix $A = [a_{ij}]_{m \times n}$ by changing its rows into columns and columns of A into rows is called the transpose of A and is denoted by A′.

- A square matrix $a = [a_{ij}]_{n \times n}$ is said to be
  (i) Symmetric if $a_{ij} = a_{ji}$ for all i and j i.e. if A′ = A.
  (ii) Skew-symmetric if
  $a_{ij} = -a_{ji}$ for all i and j i.e., if A′ = –A.

- Every square matrix A can be uniquely written as sum of a symmetric and a skew-symmetric matrix.
  $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$ where $\frac{1}{2}(A + A')$ is symmetric and $\frac{1}{2}(A - A')$ is skew-symmetric.

- Let $A = [a_{ij}]_{m \times n}$ be a given matrix. Then the matrix obtained from A by replacing all the elements by their conjugate complex is called the conjugate of the matrix A and is denoted by $\overline{A} = [\overline{a}_{ij}]$.

Properties :

(i) $\overline{(\overline{A})} = A$

(ii) $\overline{(A+B)} = \overline{A} + \overline{B}$

(iii) $\overline{(\lambda A)} = \overline{\lambda}\ \overline{A}$, where λ is a scalar

(iv) $\overline{(AB)} = \overline{A}\ \overline{B}$.

Determinant :

Consider the set of linear equations $a_1x + b_1y = 0$ and $a_2x + b_2y = 0$, where on eliminating x and y we get the eliminant $a_1b_2 - a_2b_1 = 0$; or symbolically, we write in the determinant notation

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv a_1b_2 - a_2b_1 = 0$$

Here the scalar $a_1b_2 - a_2b_1$ is said to be the expansion of the 2 × 2 order determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ having 2 rows and 2 columns.

Similarly, a determinant of 3 × 3 order can be expanded as :

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$

$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$

$= \Sigma(\pm a_ib_jc_k)$

- To every square matrix $A = [a_{ij}]_{m \times n}$ is associated a number of function called the determinant of A and is denoted by | A | or det A.

  Thus, $|A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$

- If $A = [a_{ij}]_{n \times n}$, then the matrix obtained from A after deleting ith row and jth column is called a submatrix of A. The determinant of this submatrix is called a minor or a_{ij}.
- Sum of products of elements of a row (or column) in a det with their corresponding cofactors is equal to the value of the determinant.

  i.e., $\sum_{i=1}^{n} a_{ij} C_{ij} = |A|$ and $\sum_{j=1}^{n} a_{ij} C_{ij} = |A|$.

- (i) If all the elements of any two rows or two columns of a determinant ate either identical or proportional, then the determinant is zero.

  (ii) If A is a square matrix of order n, then $|kA| = k^n |A|$.

  (iii) If Δ is determinant of order n and Δ' is the determinant obtained from Δ by replacing the elements by the corresponding cofactors, then $\Delta' = \Delta^{n-1}$

  (iv) Determinant of a skew-symmetric matrix of odd order is always zero.

- The determinant of a square matrix can be evaluated by expanding from any row or column.
- If $A = [a_{ij}]_{n \times n}$ is a square matrix and C_{ij} is the cofactor of a_{ij} in A, then the transpose of the matrix obtained from A after replacing each element by the corresponding cofactor is called the adjoint of A and is denoted by adj. A.

  Thus, adj. A = $[C_{ij}]'$.

  Properties of adjoint of a square matrix

  (i) If A is a square matrix of order n, then $A \cdot (\text{adj. } A) = (\text{adj. } A) A = |A| \; I_n$.

  (ii) If | A | = 0, then A (adj. A) = (adj. A) A = O.

  (iii) $|\text{adj. } A| = |A|^{n-1}$ if $|A| \neq 0$

  (iv) adj. (AB) = (adj. B) (adj. A).

  (v) adj. (adj. A) = $|A|^{n-2}$ A.

- Let A be a square matrix of order n. Then the inverse of A is given by $A^{-1} = \frac{1}{|A|}$ adj. A.
- Reversal law : If A, B, C are invertible matrices of same order, then

  (i) $(AB)^{-1} = B^{-1} A^{-1}$

  (ii) $(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$

- Criterion of consistency of a system of linear equations

  (i) The non-homogeneous system AX = B, B ≠ 0 has unique solution if | A | ≠ 0 and the unique solution is given by $X = A^{-1}B$.

  (ii) Cramer's Rule : If | A | ≠ 0 and $X = (x_1, x_2, \dots, x_n)'$ then for each i =1, 2, 3, …, n ; $x_i = \frac{|A_i|}{|A|}$ where Ai is the matrix obtained from A by replacing the ith column with B.

  (iii) If | A | = 0 and (adj. A) B = O, then the system AX = B is consistent and has infinitely many solutions.

  (iv) If | A | = 0 and (adj. A) B ≠ O, then the system AX = B is inconsistent.

  (v) If | A | ≠ 0 then the homogeneous system AX = O has only null solution or trivial solution (i.e., $x_1 = 0, x_2 = 0, \dots x_n = 0$)

  (vi) If | A | = 0, then the system AX = O has non-null solution.

- (i) Area of a triangle having vertices at (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

  (ii) Three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear iff area of $\Delta ABC = 0$.

- A square matrix A is called an orthogonal matrix if AA′ = AA′ = I.
- A square matrix A is called unitary if $AA^\theta = A^\theta A = I$

  (i) The determinant of a unitary matrix is of modulus unity.

  (ii) If A is a unitary matrix then A', $\overline{A}$, A^θ, A^{-1} are unitary.

  (iii) Product of two unitary matrices is unitary.

- Differentiation of Determinants :

  Let $A = |C_1 \; C_2 \; C_3|$ is a determinant then

  $$\frac{dA}{dx} = |C'_1 \; C_2 \; C_3| + |C_1 \; C'_2 \; C_3| + |C_1 \; C_2 \; C'_3|$$

  Same process we have for row.

  Thus, to differentiate a determinant, we differentiate one column (or row) at a time, keeping others unchanged.