DIFFERENTIATION

Differentiation and Applications of Derivatives :

- If y = f(x), then

1. $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$
2. $\left(\frac{dy}{dx}\right)_{x=a} = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$
3. $\left(\frac{dy}{dx}\right)_{x=a} = \lim_{x \to h} \frac{f(a+h)-f(a)}{h}$

- If u = f(x), v = ϕ(x), then

1. $\frac{d}{dx}(k) = 0$
2. $\frac{d}{dx}(ku) = k\frac{du}{dx}$
3. $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$
4. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
5. $\frac{du}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
6. If x = f(t), y = ϕ (t), then $\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt}$
7. If y = f[ϕ(x)], then $\frac{dy}{dx} = f'[\phi(x)].\ \frac{d}{dx}[\phi(x)]$
8. If w = f(y), then $\frac{dw}{dx} = f'(y)\ \frac{dy}{dx}$
9. If y = f(x), z = ϕ(x), then $\frac{dy}{dz} = \frac{dy}{dx}.\frac{dx}{dz}$
10. $\frac{dy}{dx}.\frac{dx}{dy} = 1$ or $\frac{dy}{dx} = \frac{1}{dx/dy}$

- 1. $\frac{d}{dx}(k) = 0$
2. $\frac{d}{dx}x^n = nx^{n-1}$
3. $\frac{d}{dx}\frac{1}{x^n} = -\frac{n}{x^{n+1}}$
4. $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$
5. $\frac{d}{dx}e^x = e^x$
6. $\frac{d}{dx}a^x = a^x \log a$
7. $\frac{d}{dx}\log x = \frac{1}{x}$
8. $\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e$
9. $\frac{d}{dx}\sin x = \cos x$
10. $\frac{d}{dx}\cos x = -\sin x$
11. $\frac{d}{dx}\tan x = \sec^2 x$
12. $\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$
13. $\frac{d}{dx}\sec x = \sec x \tan x$
14. $\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$
15. $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$
16. $\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$
17. $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$
18. $\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$
19. $\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$
20. $\frac{d}{dx}\operatorname{cosec}^{-1}x = -\frac{1}{x\sqrt{x^2-1}}$

- **Suitable substitutions :** The functions any also be reduced to simplar forms by the substitutions as follows.

  1. If the function involve the term $\sqrt{(a^2 - x^2)}$, then put x = a sin θ or x = a cos θ.
  2. If the function involve the term $\sqrt{(a^2 + x^2)}$, then put x = a tan θ or x = a cot θ.
  3. If the function involve the term $\sqrt{(x^2 - a^2)}$, then put x = a sec θ or x = a cosec θ.
  4. If the function involve the term $\sqrt{\frac{a-x}{a+x}}$, then put x = a cos θ or x = a cos 2θ

  All the above substitutions are also true, if a = 1

- **Differentiation by taking logarithm :**

  Differentiation of the functions of the following types are obtained by taking logarithm.

  1. When the functions consists of the product and quotient of a number of functions.
  2. When a function of x is raised to a power which is itself a function of x.

  For example, let $y = [f(x)]^{\phi(x)}$

  Taking logarithm of both sides, log y = ϕ(x) log f(x)

  Differentiating both sides w.r.t 'x',

  $$\frac{1}{y}\frac{dy}{dx} = \phi'(x)\log f(x) + \phi(x).\frac{f'(x)}{f(x)}$$

  $$= [f(x)]^{\phi(x)} \log f(x).\phi'(x) + \phi(x) . [f(x)^{\phi(x)-1}.f'(x)$$

  $\frac{dy}{dx}$ = Differential of y treading f(x) as constant + Differential of y treating ϕ(x) as constant.

  It is an important formula.

- **Differentiation of implicit functions :**

  1. If f(x, y) = 0 is a implicit function, then

  $$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

  $$= -\frac{\text{Diff. of f w.r.t. x keeping y constant}}{\text{Diff. of f w.r.t. y keeping x constant}}$$

  For example, consider $f(x, y) = x^2 + 3xy + y^2 = 0$, then

  $$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{2x+3y}{3x+2y}$$

  1. If y = f(x), then

  $$\frac{dy}{dx} = y_1 = f'(x), \quad \frac{d^2y}{dx^2} = y_2 = f''(x), ...$$

  $$\frac{d^2y}{dx^n} = y_n = f^n(x)$$

  2. $\frac{d^n}{dx^n}(ax+b)^n = n!\,a^n$
  3. $\frac{d^n}{dx^n}(ax+b)^m = m(m-1)$ .... $(m-n+1)\,a^n(ax+b)^{m-n}$
  4. $\frac{d^n}{dx^n}e^{mx} = m^n e^{mx}$
  5. $\frac{d^n}{dx^n}a^{mx} = m^n a^{mx}(\log a)^n$
  6. $\frac{d^n}{dx^n}\log(ax+b) = \frac{(-1)^{n-1}a^n(n-1)!}{(ax+b)^n}$
  7. $\frac{d^n}{dx^n}\sin(ax+b) = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$
  8. $\frac{d^n}{dx^n}\cos(ax+b) = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$

- **Leibnitz's theorem :** If u and v are any two functions of x such that their desired differential coefficients exist, then the n^{th} differential coefficient of uv is given by

  $$D^n(uv) = (D^n u)v + {}^nC_1(D^{n-1}u)(Dv) + {}^nC_2(D^{n-2}u)(D^2v) + ... + u(D^n v)$$