FUNCTIONS

Definition of a Function :

Let A and B be two sets and f be a rule under which every element of A is associated to a unique element of B. Then such a rule f is called a function from A to B and symbolically it is expressed as

$$f : A \to B$$

or $A \xrightarrow{f} B$

Function as a Set of Ordered Pairs

Every function $f : A \to B$ can be considered as a set of ordered pairs in which first element is an element of A and second is the image of the first element. Thus

$$f = \{a, f(a) / a \in A, f(a) \in B\}.$$

Domain, Codomain and Range of a Function :

If $f : A \to B$ is a function, then A is called domain of f and B is called codomain of f. Also the set of all images of elements of A is called the range of f and it is expressed by f(A). Thus

$$f(A) = \{f(a) | a \in A\}$$

obviously $f(A) \subset B$.

Note : Generally we denote domain of a function f by D_f and its range by R_f.

Equal Functions :

Two functions f and g are said to be equal functions if

- domain of f = domain of g
- codomain of f = codomain of g
- $f(x) = g(x) \ \forall \ x$.

Algebra of Functions :

If f and g are two functions then their sum, difference, product, quotient and composite are denoted by

$f + g$, $f - g$, fg, f/g, fog

and they are defined as follows :

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)\, f(g)$$

$$(f/g)(x) = f(x)/g(x) \quad (g(x) \neq 0)$$

$$(fog)(x) = f[g(x)]$$

Formulae for domain of functions :

- $D_{f \pm g} = D_f \cap D_g$
- $D_{fg} = D_f \cap D_g$
- $D_{f/g} = D_f \cap D_g \cap \{x | g(x) \neq 0\}$
- $D_{gof} = \{x \in D_f \mid f(x) \in D_g\}$
- $D_{\sqrt{f}} = D_f \cap \{x | f(x) \geq 0\}$

Classification of Functions

1. Algebraic and Transcendental Functions :

- **Algebraic functions :** If the rule of the function consists of sum, difference, product, power or roots of a variable, then it is called an algebraic function.
- **Transcendental Functions :** Those functions which are not algebraic are named as transcendental or non algebraic functions.

2. Even and Odd Functions :

- **Even functions :** If by replacing x by –x in f(x) there in no change in the rule then f(x) is called an even function. Thus

  $f(x)$ is even $\Leftrightarrow f(-x) = f(x)$

- **Odd function :** If by replacing x by –x in f(x) there is only change of sign of f(x) then f(x) is called an odd function. Thus

  f(x) is odd $\Leftrightarrow f(-x) = -f(x)$

3. **Explicit and Implicit Functions :**

- **Explicit function :** A function is said to be explicit if its rule is directly expressed (or can be expressed(in terms of the independent variable. Such a function is generally written as

  $y = f(x)$, $x = g(y)$ etc.

- **Implicit function :** A function is said to be implicit if its rule cannot be expressed directly in terms of the independent variable. Symbolically we write such a function as

  $f(x, y) = 0$, $\phi(x, y) = 0$ etc.

4. **Continuous and Discontinuous Functions :**

- **Continuous functions :** A functions is said to be continuous if its graph is continuous i.e. there is no gap or break or jump in the graph.

- **Discontinuous Functions** : A function is said to be discontinuous if it has a gap or break in its graph atleast at one point. Thus a function which is not continuous is named as discontinuous.

5. **Increasing and Decreasing Functions :**

- **Increasing Functions :** A function f(x) is said to be increasing function if for any x_1, x_2 of its domain

  $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$

  or $x_1 > x_2 \Rightarrow f(x_1) \ge f(x_2)$

- **Decreasing Functions :** A function f(x) is said to be decreasing function if for any x_1, x_2 of its domain

  $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$

  or $x_1 > x_2 \Rightarrow f(x_1) \le f(x_2)$

Periodic Functions :

A functions f(x) is called a periodic function if there exists a positive real number T such that

$$f(x + T) = f(x). \qquad \forall x$$

Also then the least value of T is called the period of the function f(x).

Period of f(x) = T

$\Rightarrow$ Period of f(nx + a) = T/n

Periods of some functions :

Function	Period
sin x, cos x, sec x, cosec x,	2π
tan x, cot x	π
$\sin^n x$, $\cos^n x$, $\sec^n x$, $\text{cosec}^n x$	2π if n is odd π if n is even
$\tan^n x$, $\cot^n x$	$\pi \ \forall\ n \in N$
$\lvert\sin x\rvert$, $\lvert\cos x\rvert$, $\lvert\sec x\rvert$, $\lvert\text{cosec } x\rvert$	π
$\lvert\tan x\rvert$, $\lvert\cot x\rvert$,	π
$\lvert\sin x\rvert + \lvert\cos x\rvert$, $\sin^4 x + \cos^4 x$ $\lvert\sec x\rvert + \lvert\text{cosec } x\rvert$	$\frac{\pi}{2}$
$\lvert\tan x\rvert + \lvert\cot x\rvert$	$\frac{\pi}{2}$
x – [x]	1
• Period of f(x) = T $\Rightarrow$ period of f(ax + b) = T/|a|	
• Period of $f_1(x) = T_1$, period fo $f_2(x) = T_2$ $\Rightarrow$ period of a $f_1(x) + bf_2(x) \le$ LCM $\{T_1, T_2\}$	

Kinds of Functions :

- **One-one/ May one Functions :**

  A function $f : A \to B$ is said to be one-one if different elements of A have their different images in B.

  Thus

$$f \text{ is one-one} \Leftrightarrow \begin{cases} a \ne b & \Rightarrow \quad f(a) \ne f(b) \\ & \text{or} \\ f(a) = f(b) & \Rightarrow \quad a = b \end{cases}$$

  A function which is not one-one is called many one. Thus if f is many one then atleast two different elements have same f-image.

- **Onto/Into Functions :** A function $f : A \to B$ is said to be onto if range of f = codomain of f

  Thus f is onto $\Leftrightarrow f(A) = B$

Hence $f : A \to B$ is onto if every element of B (co-domain) has its f–preimage in A (domain).

A function which is not onto is named as into function. Thus $f : A \to B$ is into if $f(A) \neq B$. i.e., if there exists atleast one element in codomain of f which has no preimage in domain.

Note :

Total number of functions : If A and B are finite sets containing m and n elements respectively, then

- total number of functions which can be defined from A to B = n^m.
- total number of one-one functions from A to B

$$= \begin{cases} {}^n p_m & \text{if} \quad m \leq n \\ 0 & \text{if} \quad m > n \end{cases}$$

- total number of onto functions from A to B (if $m \geq n$) = total number of different n groups of m elements.

Composite of Functions :

Let $f : A \to B$ and $g : B \to C$ be two functions, then the composite of the functions f and g denoted by gof, is a function from A to C given by $gof : A \to C$, $(gof)(x) = g[f(x)]$.

Properties of Composite Function :

The following properties of composite functions can easily be established.

- Composite of functions is not commutative i.e.,

$$fog \neq gof$$

- Composite of functions is associative i.e.

$$(fog)oh = fo(goh)$$

- Composite of two bijections is also a bijection.

Inverse Function :

If $f : A \to B$ is one-one onto, then the inverse of f i.e., f^{-1} is a function from B to A under which every $b \in B$ is associated to that $a \in A$ for which $f(a) = b$.

Thus $\quad f^{-1} : B \to A$,

$$f^{-1}(b) = a \Leftrightarrow f(a) = b.$$

Domain and Range of some standard functions :

Function	Domain	Range
Polynomial function	R	R
Identity function x	R	R
Constant function c	R	{c}
Reciprocal function 1/x	R_0	R_0
x^2, $\lvert x\rvert$	R	$R^+ \cup \{0\}$
x^3, $x\,\lvert x\rvert$	R	R
Signum function	R	{–1, 0, 1}
$x + \lvert x\rvert$	R	$R^+ \cup \{0\}$
$x - \lvert x\rvert$	R	$R^- \cup \{0\}$
[x]	R	Z
x – [x]	R	[0, 1)
$\sqrt{x}$	$[0, \infty)$	$[0, \infty)$
a^x	R	R^+
log x	R^+	R
sin x	R	[–1, 1]
cos x	R	[–1, 7]
tan x	$R - \{\pm \pi/2, \pm 3\pi/2, ...\}$	R
cot x	$R - \{0, \pm \pi. \pm 2\pi, ...$	R
sec x	$R - (\pm \pi/2, \pm 3\pi/2, ...$	R – (–1, 1)
cosec x	$R - \{0, \pm\pi, \pm 2\pi, ...\}$	R –(–1, 1)
sinh x	R	R
cosh x	R	$[1, \infty)$
tanh x	R	(–1, 1)
coth x	R_0	R –[1, –1]
sech x	R	(0, 1]
cosech x	R_0	R_0
$\sin^{-1} x$	[–1, 1]	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	[–1, 1]	$[0, \pi]$
$\tan^{-1} x$	R	$(-\pi/2, \pi/2\}$
$\cot^{-1} x$	R	$(0, \pi)$
$\sec^{-1} x$	R –(–1, 1)	$[0, \pi] - \{\pi/2\}$
$\text{cosec}^{-1} x$	R – (–1, 1)	$(-\pi/2, \pi/2] - \{0\}$