COMPLEX NUMBERS

- $\sqrt{-1}$ is denoted by 'i' and is pronounced as 'iota'.
  $i = \sqrt{-1} \Rightarrow i^2 = -1, i^3 = -i, i^4 = 1$.
- If a, b $\in$ R and $i = \sqrt{-1}$ then a + ib is called a complex number. The complex number a + ib is also denoted by the ordered pair (a, b)
- If z = a + ib is a complex number, then :
  (i) a is called the real part of z and we write
  Re (z) = a.
  (ii) b is called the imaginary part of z and we write
  Im (z) = b
- Two complex numbers z_1 and z_2 are said to be equal complex numbers if Re (z_1) = Re (z_2) and Im (z_1) = Im (z_2).
- If z = x + iy is a non zero complex number, then 1/z is called the multiplicative inverse of z.
- If x + iy is a complex number, then the complex number x – iy is called the conjugate of the complex number x + iy and we write $\overline{x+iy} = x - iy$.
- **Algebra of Complex Numbers**
  (i) **Addition :** (a + ib) + (c + id) = (a + c) + i(b + d)
  (ii) **Subtraction :**
  (a + ib) – (c + id) = (a – c) + i(b – d)
  (iii) Multiplication :
  (a + ib) + (c + id) = (ac – bd) + i(ab + bc)
  (iv) **Division by a non-zero complex number :**
  $$\frac{a+ib}{c+id} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}, (c + id) \neq 0$$
- **Properties** : If z_1, z_2 are complex numbers, then
  (i) $\overline{(\overline{z_1})} = z_1$
  (ii) $z + \overline{z} = 2$ Re (z)
  (iii) $z - \overline{z} = 2i$ Im (z)
  (iv) $z = \overline{z}$ iff z is purely real
  (v) $z = \overline{z}$ iff z is purely imaginary
  (vi) $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$
  (vii) $\overline{z_1 - z_2} = \overline{z}_1 - \overline{z}_2$
  (viii) $\overline{z_1 . z_2} = \overline{z}_1 . \overline{z}_2$
  (ix) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}$ provided $z_2 \neq 0$
- If x + iy is a complex number, then the non-negative ral number $\sqrt{x^2+y^2}$ is called the modulus of the complex number x + iy and write
  $|x + iy| = \sqrt{x^2+y^2}$
  Properties : If z_1, z_2 are complex numbers, then
  (i) $|z_1| = 0$ iff $z_1 = 0$
  (ii) $|z_1| = |\overline{z}_1| = |-z_1|$
  (iii) $-|z_1| \leq$ Re $(z_1) \leq |z_1|$
  (iv) $-|z_1| \leq$ Im $(z_1) \leq |z_1|$
  (v) $|z_1 \overline{z}_1| = |z_1|^2$
  (vi) $|z_1 + z_2| \leq |z_1| + |z_2|$
  (vii) $|z_1 - z_2| \geq |z_1| - |z_2|$
  (viii) $|z_1 z_2| = |z_1| |z_2|$
  (ix) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, provided $z_2 \neq 0$
  (x) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2$ Re $(z_1 \overline{z}_2)$
  (xi) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2$ Re $(z_1 \overline{z}_2)$
  (xi) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2 [|z_1|^2 + |z_2|^2]$.
- De Moivre's Theorem
  (i) If n is any integer (positive or negative), then
  $(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$
  (ii) If n is a rational number, then the value or one of the values of $(\cos\theta + i \sin\theta)^n$ is $\cos n\theta + i \sin n\theta$
- Euler's Formula
  $e^{i\theta} = \cos\theta + i \sin\theta$ and $e^{-i\theta} = \cos\theta - i \sin\theta$
- Square root of complex number
  Square root of z = a + ib are given by
  $$\pm\left[\sqrt{\left(\frac{|z|+a}{2}\right)} + i\sqrt{\left(\frac{|z|-a}{2}\right)}\right] \text{ for } b > 0 \text{ and}$$
  $$\pm\left[\sqrt{\left(\frac{|z|+a}{2}\right)} - i\sqrt{\left(\frac{|z|-a}{2}\right)}\right] \text{ for } b < 0.$$

- If $\omega = \frac{-1+i\sqrt{3}}{2}$, then the cube roots of unity are 1, ω and ω^2. We have:

  (i) $1 + \omega + \omega^2 = 0$ (ii) $\omega^3 = 1$
- Let $z = x + iy$ be any complex number.

  Let $z = r(\cos\theta + i\sin\theta)$ where $r > 0$.

  $\therefore x = r\cos\theta$ and $y = r\sin\theta$

  $\therefore x^2 + y^2 = r^2$

  $\Rightarrow \quad r = \sqrt{x^2+y^2} \qquad (\because r > 0)$

  $\therefore \cos\theta = \frac{x}{\sqrt{x^2+y^2}}$ and $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$

  The value of θ is found by solving these equations. θ is called the argument (or amplitude) of z.

  If $-p < \theta \le \pi$, then θ is called the principal argument of z.
- Identification of θ –

x	y	arg(z)	Interval of θ
+	+	θ	$\left(0 < \theta < \frac{\pi}{2}\right)$
+	–	$-\theta$	$\left(\frac{-\pi}{2} < \theta < 0\right)$
–	+	$(\pi - \theta)$	$\left(\frac{\pi}{2} < \theta < \pi\right)$
–	–	$-(\pi - \theta)$	$\left(-\pi < \theta < \frac{-\pi}{2}\right)$

- If z_1 and z_2 are two complex numbers then

  (i) $|z_1 - z_2|$ is the distance between the points with affixes z_1 and z_2.

  (ii) $\frac{mz_2 + nz_1}{m+n}$ is the affix of the point dividing the line joining the points with affixes z_1 and z_2 in the ratio m : n internally.

  (iii) $\frac{mz_2 - nz_1}{m-n}$ is the affix of the point dividing the line joining the points with affixes z_1 and z_2 in the ratio m : n externally where $m \ne n$.

  (iv) If z_1, z_2, z_3 are the affixes of the vertices of a triangle then the affix of its centroid is $\frac{z_1+z_2+z_3}{3}$.

  (v) $z = tz_1 + (1 - t)z_2$ is the equation of the line joining points with affixes z_1 and z_2. Here 't' is a parameter.

  (vi) $\frac{z-z_1}{z_2-z_1} = \frac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1}$ is the equation of the line joining points with affixes z_1 and z_2.
- Three points with affixes z_1, z_2, z_3 are collinear if

  $$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0.$$
- The general equation of a straight line is

  $\bar{a}z + a\bar{z} + b = 0$, where b is any real number.
- (i) $|z - z_1| < r$ represents the circle with centre z_1 and radius r.

  (ii) $|z - z_1| < r$ represents the interior of the circle with centre z_1 and radius r.
- $\left|\frac{z-z_1}{z-z_1}\right| = k$ represents a circle line which is the perpendicular bisector of the line segment joining points with affixes z_1 and z_2.
- $(z - z_1)(\bar{z}-\bar{z}_2) + (\bar{z}-\bar{z}_1) + (z - z_2) = 0$ represents the circle with line joining points with affixes z_1 and z_2 as a diameter.
- $|z - z_1| + |z - z_2| = 2k$, $k \in R^+$ represents the ellipse with foci at points with affixes z_1 and z_2.
- If z_1, z_2, z_3 be the affixes of the points A, B, C respectively, then the angle between AB and AC is given by $\arg\left(\frac{z_3-z_1}{z_2-z_1}\right)$.
- If z_1, z_2, z_3, z_4 are the affixes of the points A, B, C, D respectively, then the angle between AB and CD is given by $\arg\left(\frac{z_2-z_1}{z_4-z_3}\right)$.
- nth roots of a complex number

  Let $z = r(\cos\theta + i\sin\theta)$, $r > 0$ be any complex number. nth root o $z = z^{1/n}$

  $$= r^{1/n}\left(\cos\frac{2k\pi+\theta}{n} + i\sin\frac{2k\pi+\theta}{n}\right),$$

  where $k = 0, 1, 2, \ldots\ldots\ldots, n - 1$.

  There are n distinct values and sum of all these values is 0.
- Logarithm of a complex number

  Let $z = re^{i\theta}$ be any complex number.

  Then $\log z = \log re^{i\theta} = \log r + \log e^{i\theta}$

  $= \log r + i\theta \log e = \log r + i\theta$.

  $\therefore \quad \log z = \log|z| + i\,\text{amp}(z)$.